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The Remainder Term

If the Taylor series for a function $f(x)$ is truncated at the n^{th} term, what is the difference between $f(x)$ and the value given by the n^{th} Taylor polynomial? That is, what is the error involved in using the Taylor polynomial to approximate the function?

Theorem. Suppose you expand f around c, and that f is $(n+1)$-times continuously differentiable on an open interval containing c. If x is another point in this interval, then for some z in the open interval between x and c,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}.$$

$p_n(x;c) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$ is the n^{th} degree Taylor polynomial at c. The other term on the right is called the **Lagrange remainder term**:

$$R_n(x;c) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}.$$

The appearance of z, a point between x and c, and the fact that it's being plugged into a derivative suggest that there is a connection between this result and the Mean Value Theorem. In fact, for $n = 0$ the result says that there is a number z between c and x such that

$$f(x) = f(c) + f'(z) \cdot (x-c).$$

This is the Mean Value Theorem.

On the one hand, this reflects the fact that Taylor's theorem is proved using a generalization of the Mean Value Theorem. On the other hand, this shows that you can regard a Taylor expansion as an *extension* of the Mean Value Theorem.

Example. Compute the Remainder Term $R_3(x;1)$ for $f(x) = \sin 2x$.

For the *third* remainder term, I need the *fourth* derivative:

$$f'(x) = 2\cos 2x, \quad f''(x) = -4\sin 2x, \quad f'''(x) = -8\cos 2x, \quad f^{(4)}(x) = 16\sin 2x.$$

The Remainder Term is

$$R_3(x;1) = \frac{16\sin 2z}{4!}(x-1)^4.$$

z is a number between x and 1. □

Example. Compute the Remainder Term $R_n(x;3)$ for $f(x) = e^{4x}$.

Since I want the n^{th} Remainder Term, I need to find an expression for the $(n+1)^{\text{st}}$ derivative. I'll compute derivative until I see a pattern:

$$f'(x) = 4e^{4x}, \quad f''(x) = 4^2 e^{4x}, \quad f'''(x) = 4^3 e^{4x}.$$

Notice that it's easier to see the pattern if you don't multiply out the power of 4.

Thus,

$$f^{(n)}(x) = 4^n e^{4x}, \quad \text{so} \quad f^{(n+1)}(x) = 4^{n+1} e^{4x}.$$

The Remainder Term is

$$R_n(x;3) = \frac{4^{n+1}e^{4z}}{(n+1)!}(x-3)^{n+1}.$$

z is a number between x and 3. $\square$

There are several things you might do with the Remainder Term:

1. Estimate the error in using $p_n(x;c)$ to estimate $f(x)$ on a given interval $(c-r, c+r)$. (The interval and the degree n are fixed; you want to find the error.)

2. Find the smallest value of n for which $p_n(x;c)$ approximates $f(x)$ to within a given error ("tolerance") on a given interval $(c-r, c+r)$. (The interval and the error are fixed; you want to find the degree.)

3. Find the largest interval $(c-r, c+r)$ on which $p_n(x;c)$ approximates $f(x)$ to within a given error ("tolerance"). (The degree and the error are fixed; you want to find the interval.)

Example. The Maclaurin series for $\ln(1+x)$ is

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

What is the largest error which might result from using the first three terms of the series to approximate $\ln(1+x)$, if $0 \le x \le 1$?

The remainder term is

$$R_n(x;0) = \frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1},$$

I have $0 < z < x$. I want to estimate the maximum size of $|R_3(x;0)|$. I take absolute values, because I don't care whether the error is positive or negative, only how large it is.

$f(x) = \ln(1+x)$, and you can check by taking derivatives that $f^{(4)}(x) = \dfrac{-6}{(1+x)^4}$. Thus, $f^{(4)}(z) = \dfrac{-6}{(1+z)^4}$. So

$$|R_3(x;0)| = \left|\frac{\frac{-6}{(1+z)^4}}{4!}(x-0)^4\right| = \frac{1}{4}\frac{1}{(1+z)^4}|x|^4.$$

Since I want the largest possible error, I want to see how large the terms $\dfrac{1}{(1+z)^4}$ and $|x|^4$ could be. Remember that z is between 0 and x, and $0 \le x \le 1$. So

$$0 < z < x \le 1.$$

First, $0 \le x \le 1$ means that

$$|x|^4 \le 1^4 = 1.$$

How large can $\dfrac{1}{(1+z)^4}$ be, given that $0 < z < 1$? As z goes from 0 to 1, $\dfrac{1}{(1+z)^4}$ *decreases*, so it is largest if $z = 0$. This means that

$$\frac{1}{(1+z)^4} \le 1.$$

You can also see this by doing the algebra:

$$\begin{aligned}
&0 < z < 1 \\
&1 < z + 1 < 2 \\
&1 < (z+1)^4 < 16 \\
&1 > \frac{1}{(1+z)^4} > \frac{1}{16}
\end{aligned}$$

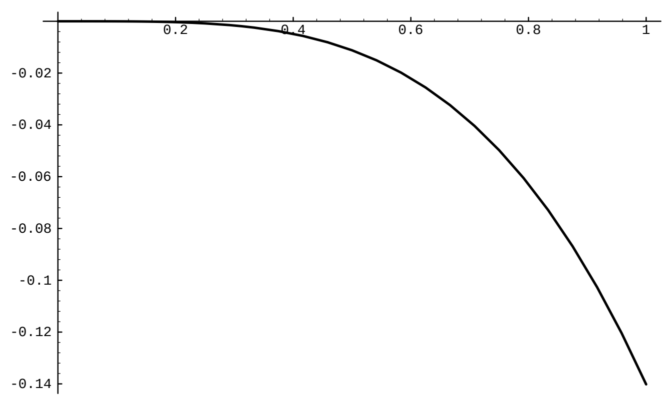
In general, to estimate the z-term you'd have to find the absolute max on the interval for z. If you know that the z-term is either increasing or decreasing, you can check its value at the interval endpoints, and take the largest.

Using the estimates for $\dfrac{1}{(1+z)^4}$ and $|x|^4$, I have

$$|R_3(x;0)| \le \frac{1}{4} \cdot 1 \cdot 1 = \frac{1}{4}.$$

The error is no greater than $\dfrac{1}{4}$.

I can check this by plotting the difference between the 3^{rd} degree Taylor polynomial and $\ln(1+x)$.



From the picture, it looks as though the maximum error is around 0.15 (in absolute value). The estimated error was pretty conservative. □

Example. (a) Compute $R_3(x;0)$ for $f(x) = \dfrac{1}{2+x}$, and express $f(x)$ using $p_3(x)$ and the remainder term.

(b) Use $R_3(x;0)$ to approximate the largest error that occurs in using $p_3(x)$ to approximate $\dfrac{1}{2+x}$ for $0 \le x \le 1$.

(a) Since I want $R_3(x;0)$, I need the fourth derivative:

$$f'(x) = \frac{-1}{(2+x)^2}, \quad f''(x) = \frac{2}{(2+x)^3}, \quad f'''(x) = \frac{-6}{(x+x)^4}, \quad f^{(4)}(x) = \frac{24}{(2+x)^5}.$$

Thus,

$$R_3(x;0) = \frac{24}{(2+z)^5} \cdot \frac{1}{4!} x^4 = \frac{x^4}{(2+z)^5}.$$

Now

$$\frac{1}{2+x} = \frac{1}{2} \cdot \frac{1}{1 - \left(-\frac{x}{2}\right)} = \frac{1}{2} \cdot \left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \cdots\right).$$

Therefore,

$$\frac{1}{2+x} = \frac{1}{2} \cdot \left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8}\right) + \frac{x^4}{(2+z)^5}.$$

Here z is between 0 and x. □

(b) I have

$$|R_3(x;0)| = \frac{1}{(2+z)^5} \cdot |x|^4.$$

I'll estimate the z and x-terms one at a time.
Since $0 \le x \le 1$, I have

$$|x|^4 \le 1^4 = 1.$$

Since $0 \le x \le 1$ and z is between 0 and x, it follows that $0 \le z \le 1$. On this interval, $\dfrac{1}{(2+z)^5}$ decreases, so it attains its largest value at $z = 0$. Therefore,

$$\frac{1}{(2+z)^5} \le \frac{1}{(2+0)^5} = \frac{1}{32}.$$

Alternatively,

$$\begin{aligned} 0 &< z < 1 \\ 2 &< 2 + z < 3 \\ 32 &< (2+z)^5 < 243 \\ \frac{1}{32} &> \frac{1}{(2+z)^5} > \frac{1}{243} \end{aligned}$$

Thus,

$$|R_3(x;0)| \le \frac{1}{32} \cdot 1 = \frac{1}{32}.$$

The error is no greater than $\dfrac{1}{32}$. □

Example. Find the smallest value of n for which the n^{th} degree Taylor series for $f(x) = e^{2x}$ at $c = 0$ approximates e^{2x} on the interval $0 \le x \le 0.3$ with an error no greater than 10^{-6}.

Notice that

$$f'(x) = 2e^{2x}, \quad f''(x) = 2^2 e^{2x}, \quad f^{(3)}(x) = 2^3 e^{2x}, \quad \ldots, \quad f^{(n)}(x) = 2^n e^{2x}.$$

So

$$|R_n(x;0)| = \left|\frac{2^{n+1}e^{2z}}{(n+1)!}x^{n+1}\right| = \frac{2^{n+1}e^{2z}}{(n+1)!}|x|^{n+1} \quad \text{for} \quad 0 \le z \le x \le 0.3.$$

First, I'll estimate how large the z and x-terms can be. Since $0 \le x \le 0.3$ and x^n is an increasing function, I have

$$|x|^{n+1} \le 0.3^{n+1}.$$

Since $0 \le z \le 0.3$ and since e^{2z} is an increasing function, I have

$$e^{2z} \le e^{0.6}.$$

Thus,

$$|R_n(x;0)| \le \frac{2^{n+1}e^{0.6}}{(n+1)!} \cdot 0.3^{n+1} = e^{0.6}\frac{0.6^{n+1}}{(n+1)!}.$$

Therefore, I want the smallest n for which

$$e^{0.6}\frac{0.6^{n+1}}{(n+1)!} < 10^{-6}.$$

I can't solve this inequality algebraically, so I'll have to use trial-and-error:

n	$e^{0.6}\dfrac{0.6^{n+1}}{(n+1)!}$
1	$0.32798\ldots$
2	$0.06559\ldots$
3	$0.00983\ldots$
4	$0.00118\ldots$
5	$1.18073\ldots \cdot 10^{-4}$
6	$1.01205\ldots \cdot 10^{-5}$
7	$7.59042\ldots \cdot 10^{-7}$

The smallest value of n is $n = 7$. □

You can also use the Remainder Term to estimate the error in using a Taylor polynomial to approximate an integral.

Example. Calvin wants to impress Phoebe Small by using the MacLaurin series for e^{2x} to approximate $\int_0^{0.5} xe^{2x}\,dx$ to within 0.0001. How many terms of the series should he use?

The Maclaurin series for e^{2x} is

$$e^{2x} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}.$$

(Substitute $u = 2x$ in the standard series for e^u.) I want to know how many terms of the series to use to approximate the integral.

Since $f(x) = e^{2x}$, I have

$$f'(x) = 2e^{2x}, \quad f''(x) = 2^2e^{2x}, \ldots f^n(x) = 2^n e^{2x}.$$

Therefore,

$$R_n(x) = \frac{1}{(n+1)!}f^{(n+1)}(z)(x-c)^{n+1} = \frac{1}{(n+1)!} \cdot 2^n \cdot e^{2z} \cdot x^{n+1}.$$

In the integral, x goes from 0 to 0.5, and z is a number between 0 (the expansion point) and x. Therefore, I know that z is a number between 0 and 0.5. Taking the worst possible case, the largest e^{2c} could be is $e^{2 \cdot 0.5} = e$. Replace e^{2z} with e to obtain

$$R_n(x) \le \frac{1}{(n+1)!} \cdot 2^n \cdot e \cdot x^{n+1}.$$

Insert this into the integral (remembering to multiply by x):

$$\text{error} \le \int_0^{0.5} \frac{1}{(n+1)!} \cdot 2^n \cdot e \cdot x^{n+2}\,dx = \frac{1}{(n+1)!} 2^{n+1} \cdot e \cdot \frac{1}{n+3} \cdot (0.5)^{n+3}.$$

I want the smallest value of n for which this ugly mess is less than 0.0001. The easiest way to do this is by trial: Plug in successive values of n.

n	$\dfrac{2^{n+1}}{(n+1)!} \cdot \dfrac{e}{n+3} \cdot 0.5^{n+3}$
0	$0.226523485\ldots$
1	$0.084946307\ldots$
2	$0.022652348\ldots$
3	$0.004719239\ldots$
4	$0.000809012\ldots$
5	$0.000117980\ldots$
6	$0.000014981\ldots$

$n = 6$ is the smallest value that works. $\square$