3-16-2019

Infinite Series

An **infinite series** is a sum

$$a_0 + a_1 + a_2 + \cdots + a_n + \cdots.$$

I can use summation notation if I don't want to write the terms out:

$$\sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + \cdots + a_n + \cdots.$$

For example,

$$\sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - 1 + 1 - \cdots + (-1)^n + \cdots,$$

$$\sum_{k=0}^{\infty} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \cdots,$$

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots.$$

Addition is not defined for an infinite collection of numbers. I have to define what I mean by the **sum** of an infinite series like those above. To do this, I'll look at the **sequence of partial sums**. For $a_0 + a_1 + a_2 + \cdots + a_n + \cdots$, the partial sums are

$$\begin{aligned} s_0 &= a_0 \\ s_1 &= a_0 + a_1 \\ s_2 &= a_0 + a_1 + a_2 \\ &\vdots \\ s_n &= a_0 + a_1 + a_2 + \cdots + a_n \end{aligned}$$

To say that the sum of the series is S means that the sequence of partial sums converges to S:

$$\lim_{n \to \infty} s_n = S.$$

The notation for this is

$$\sum_{k=0}^{\infty} a_k = S.$$

It is often difficult to compute the sum of an infinite series exactly. However, you can often tell that a series converges without knowing what it converges to. If necessary, a computer can be used to approximate the sum of a convergent series.

Some infinite series are already familiar to you. For example, the decimal representation of a real number is a convergent infinite series. Here is the number π:

$$3.14159265\ldots 3 + \frac{1}{10} + \frac{4}{100} + \frac{1}{1000} + \frac{5}{10000} + \cdots.$$

It's an infinite non-repeating decimal.

Repeating decimals represent rational numbers. I'll show by example how to convert a repeating decimal to a rational fraction. Consider $0.272727\ldots$. Set $x = 0.272727\ldots$. Then

$$\begin{aligned} 100x &= 27.272727\ldots \\ x &= 0.272727\ldots \\ \hline 99x &= 27 \\ x &= \frac{27}{99} = \frac{3}{11} \end{aligned}$$

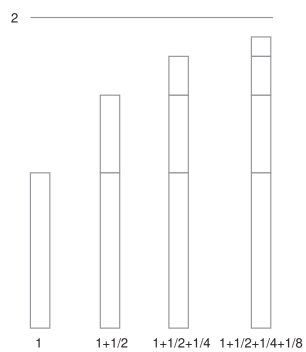
Definition. A **geometric series** is a series of the form

$$a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots = \sum_{k=0}^{\infty} ar^k.$$

The picture below shows the partial sums of the geometric series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots.$$

Notice that the partial sums seem to approach 2.



To find a formula for the sum of a geometric series, I'll start by computing the n^{th} partial sum. By long division,

$$\frac{1}{1-r} = 1 + r + r^2 + \cdots + r^n + \frac{r^{n+1}}{1-r}.$$

(This will make sense provided that $r \neq 1$.)

Multiply by a, then move the last term on the right to the left:

$$\sum_{k=0}^{n} ar^k = a + ar + ar^2 + ar^3 + \cdots + ar^n = \frac{a}{1-r} - \frac{ar^{n+1}}{1-r} = \frac{a - ar^{n+1}}{1-r}.$$

This gives a formula for the sum of a *finite* geometric series.

For instance,

$$3 + 3 \cdot 5 + 3 \cdot 5^2 + \cdots + 3 \cdot 5^{100} = \frac{3}{1-5} - \frac{3 \cdot 5^{101}}{1-5} = \frac{3}{4}\left(5^{101} - 1\right).$$

What about the infinite series

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots ?$$

The series converges if the limit of n^{th} partial sum exists. I need to compute

$$\lim_{n\to\infty}\left(\frac{a}{1-r}-\frac{ar^{n+1}}{1-r}\right)=\frac{a}{1-r}-\frac{a}{1-r}\lim_{n\to\infty}r^{n+1}.$$

By a result on geometric sequences,

$$\lim_{n\to\infty}r^{n+1}=\begin{cases}0 & \text{if } |r|<1\\ \text{diverges} & \text{if } |r|\geq 1\end{cases}.$$

Hence,

$$\sum_{k=0}^{\infty}ar^k=\begin{cases}\dfrac{a}{1-r} & \text{if } |r|<1\\ \text{diverges} & \text{if } |r|\geq 1\end{cases}.$$

For instance,

$$\sum_{k=0}^{\infty}\frac{1}{2^k}=1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}+\cdots=\frac{1}{1-\dfrac{1}{2}}=2,$$

$$\sum_{k=0}^{\infty}3\cdot 5^k=3+3\cdot 5+3\cdot 5^2+\cdots+3\cdot 5^n+\cdots\ \text{diverges.}$$

What about something like

$$\sum_{k=5}^{\infty}4\cdot\left(-\frac{1}{3}\right)^k=4\cdot\left(-\frac{1}{3}\right)^5+4\cdot\left(-\frac{1}{3}\right)^6+4\cdot\left(-\frac{1}{3}\right)^7+\cdots?$$

Well, this is

$$4\cdot\left(-\frac{1}{3}\right)^5\cdot 1+4\cdot\left(-\frac{1}{3}\right)^5\cdot\left(-\frac{1}{3}\right)+4\cdot\left(-\frac{1}{3}\right)^5\cdot\left(-\frac{1}{3}\right)^2+\cdots=\frac{4\cdot\left(-\dfrac{1}{3}\right)^5}{1-\left(-\dfrac{1}{3}\right)}=-\frac{1}{81}.$$

Example. (Retirement) \$200 is deposited each month and collects 4.8% annual interest, compounded monthly. How much is in the account after 30 years?

Note that 30 years is 360 months.

4.8% annual interest, compounded monthly, means that each month the amount in the account earns $\dfrac{4.8\%}{12}=0.4\%$ interest. This means that the amount in the account is multiplied by 1.004 each month.

The table below tracks each monthly deposit. The first row represents the first \$200 deposited, the second row the second \$200 deposited, and so on.

1 month	2 months	3 months	...	360 months
200	$1.004\cdot 200$	$1.004^2\cdot 200$	...	$1.004^{359}\cdot 200$
	200	$1.004\cdot 200$	...	$1.004^{358}\cdot 200$
		200	...	$1.004^{357}\cdot 200$
				⋮
			200	$1.004^1\cdot 200$
				200

The total amount in the account is the sum of the numbers in the last column, which is

$$200 \cdot \sum_{n=0}^{359} 1.004^n = 200 \cdot \frac{1 - 1.004^{360}}{1 - 1.004} \approx 160429.$$

By comparison, with *no* interest — e.g. if you put \$200 a month under your mattress — you'd only have \$72000 after 30 years.

At the same time, this is a rather sobering conclusion. Many people would find it a challenge to put away \$200 a month toward retirement. This problem shows that doing so and assuming a moderate interest rate produces a significant total — but definitely not enough to retire on! □

The **harmonic series** is the series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots.$$

It is an important example of a divergent series.

Here's an easy way to convince yourself that it diverges. Write down the terms of the series, and underneath the terms of a new series. The terms of the new series are all less than or equal to the terms of the harmonic series:

$$\begin{array}{ccccccccccccccccc} 1 & + & \frac{1}{2} & + & \frac{1}{3} & + & \frac{1}{4} & + & \frac{1}{5} & + & \frac{1}{6} & + & \frac{1}{7} & + & \frac{1}{8} & + & \cdots \\ \frac{1}{2} & + & \frac{1}{2} & + & \frac{1}{4} & + & \frac{1}{4} & + & \frac{1}{8} & + & \frac{1}{8} & + & \frac{1}{8} & + & \frac{1}{8} & + & \cdots \end{array}$$

Do you see the pattern? Next, you'll take the 8 terms of the harmonic series from $\frac{1}{9}$ to $\frac{1}{16}$ and write $\frac{1}{16}$ under each of them, then the next 16 terms of the harmonic series, and so on.

Consider the series on the bottom. The first two terms are both $\frac{1}{2}$. After that, the sum of the next two terms is is $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. After that, the sum of the next four terms is

$$\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}.$$

And so on. So the series on the bottom is an infinite sum of $\frac{1}{2}$'s, which goes to infinity. Since term-by-term the harmonic series is at least as big, it must diverge to infinity as well.

The harmonic series is a member of a family of series called **p-series**. Here the the facts about convergence and divergence of p-series.

Proposition. Let $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ be a p-series, where $p > 0$.

(a) If $p > 1$, the series converges.

(b) If $0 < p < 1$, the series diverges.

The harmonic series is the case $p = 1$. For example,

$$\sum_{k=1}^{\infty} \frac{1}{k^3} \quad \text{converges},$$

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \quad \text{diverges,}$$

$$\sum_{k=1}^{\infty} k^{-3/4} \quad \text{diverges.}$$

I'll prove the result above using the **Integral Test**, which I'll discuss later.

If $p > 1$, the sum of the p-series is denoted $\zeta(p)$. Thus,

$$\zeta(3) = \sum_{k=1}^{\infty} \frac{1}{k^3}.$$

It isn't too difficult to find closed form expressions for $\zeta(2n)$, where n is an integer. For instance,

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

However, the odd sums $\zeta(2n+1)$ are somewhat mysterious. It was only in 1978 that R. Apéry showed that $\zeta(3)$ is irrational. No one knows what its exact value is, and no one knows if (for instance) $\zeta(5)$ is irrational.

Here are some properties of convergent and divergent series.

Proposition. (a) If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge, then $\sum_{k=1}^{\infty}(a_k + b_k)$ converges, and

$$\sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k.$$

(b) If $\sum_{k=1}^{\infty} a_k$ converges and c is a constant, then $\sum_{k=1}^{\infty} c \cdot a_k$ converges, and

$$\sum_{k=1}^{\infty} c \cdot a_k = c \cdot \sum_{k=1}^{\infty} a_k.$$

(c) If $\sum_{k=m}^{\infty} a_k$ converges and $n \ge m$, then $\sum_{k=n}^{\infty} a_k$ converges. Likewise, if $\sum_{k=m}^{\infty} a_k$ diverges and $n \ge m$, then $\sum_{k=n}^{\infty} a_k$ diverges.

(d) If $\sum_{k=1}^{\infty} a_k$ converges and $\sum_{k=1}^{\infty} b_k$ diverges, then $\sum_{k=1}^{\infty}(a_k + b_k)$ diverges.

Proof. I'll sketch a proof of (a). I know $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge. Let s_n denote the n^{th} partial sum of $\sum_{k=1}^{\infty} a_k$ and let t_n denote the n^{th} partial sum of $\sum_{k=1}^{\infty} b_k$. Then if $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, then

$$\lim_{n\to\infty} s_n = A \quad \text{and} \quad \lim_{t\to\infty} b_n = B.$$

Now

$$s_n + t_n = (a_1 + a_2 + \cdots + a_n) + (b_1 + b_2 + \cdots b_n) = (a_1 + b_1) + (a_2 + b_2) + \cdots (a_n + b_n).$$

Consider

$$u_n = (a_1 + b_1) + (a_2 + b_2) + \cdots (a_n + b_n).$$

This is the n^{th} partial sum of $\displaystyle\sum_{k=1}^{\infty}(a_k + b_k)$. But my results on limits of sequences show that

$$\lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n = \lim_{n\to\infty}(s_n + t_n) = \lim_{n\to\infty} u_n.$$

In other words,

$$\sum_{k=1}^{\infty}(a_k + b_k) = \lim_{n\to\infty} u_n = A + B = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k. \quad \square$$

Remark. If $\displaystyle\sum_{k=1}^{\infty} a_k$ and $\displaystyle\sum_{k=1}^{\infty} b_k$ both diverge, it can happen that $\displaystyle\sum_{k=1}^{\infty}(a_k + b_k)$ converges. For example,

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{and} \quad \sum_{n=1}^{\infty} \left(-\frac{1}{n}\right) \quad \text{diverge.}$$

But the sum series is

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n}\right) = \sum_{n=1}^{\infty} 0 = 0. \quad \square$$

Example. Compute $\displaystyle\sum_{k=0}^{\infty} \frac{2^n + 3^n}{4^n}$.

By the results on geometric series, I have

$$\sum_{k=0}^{\infty} \frac{2^n}{4^n} = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1 - \frac{1}{2}} = 2,$$

$$\sum_{k=0}^{\infty} \frac{3^n}{4^n} = \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^n = \frac{1}{1 - \frac{3}{4}} = 4.$$

Therefore,

$$\sum_{k=0}^{\infty} \frac{2^n + 3^n}{4^n} = \sum_{k=0}^{\infty} \frac{2^n}{4^n} + \sum_{k=0}^{\infty} \frac{3^n}{4^n} = 2 + 4 = 6. \quad \square$$

Example. Compute

$$\frac{7}{3} + \frac{7}{3^2} + \frac{7}{3^3} + \cdots + \frac{7}{3^n} + \cdots.$$

In summation form, this is $\displaystyle\sum_{k=1}^{\infty} \frac{7}{3^k}$. I have

$$\sum_{k=1}^{\infty} \frac{7}{3^k} = \frac{7}{3} \sum_{k=1}^{\infty} \frac{1}{3^{k-1}} = \frac{7}{3} \sum_{k=0}^{\infty} \frac{1}{3^k} = \frac{7}{3} \cdot \frac{1}{1 - \frac{1}{3}} = \frac{7}{3} \cdot \frac{3}{2} = \frac{7}{2}.$$

In the second equality, I just renumbered the terms. I can do this because both $\displaystyle\sum_{k=1}^{\infty} \frac{1}{3^{k-1}}$ and $\displaystyle\sum_{k=0}^{\infty} \frac{1}{3^k}$ represent the same series, namely

$$1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots. \quad \square$$

Sometimes it's possible to use algebraic tricks to find the sum of a convergent series. The trick in the next example uses partial fractions; it's called **telescoping** because of the way the terms end up cancelling in pairs.

Example. **(Telescoping series)** Find $\displaystyle\sum_{k=0}^{\infty} \frac{2}{k^2 + 6k + 8}$.

By partial fractions,

$$\frac{2}{k^2 + 6k + 8} = \frac{1}{k+2} - \frac{1}{k+4}.$$

Then

$$\sum_{k=0}^{\infty} \left(\frac{1}{k+2} - \frac{1}{k+4}\right) = \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{6} - \frac{1}{8}\right) + \cdots.$$

All of the fractions except for $\frac{1}{2}$ and $\frac{1}{3}$ cancel. Hence,

$$\sum_{k=0}^{\infty} \frac{2}{k^2 + 6k + 8} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}. \quad \square$$

In many cases, it can be very difficult to find the **sum** of a series. Still, it's useful to be able to tell whether a series converges or diverges. If the series converges, you can use a computer (say) to approximate the sum as closely as you want.

For this reason, this discussion of infinite series will continue by looking at **tests for convergence or divergence**. The first test, the **Zero Limit Test**, can be used to tell whether a series diverges.

Theorem. (**Zero Limit Test**) If the series $\displaystyle\sum_{k=1}^{\infty} a_k$ converges, the terms of the series must go to 0.

Proof. Suppose that $\displaystyle\sum_{k=1}^{\infty} a_k$ converges to a sum S. I want to show that

$$\lim_{k \to \infty} a_k = 0.$$

The definition of the limit says that I have to show that I can make the a_k's lie within any tolerance ϵ of 0 by making the k's big enough. (Remember that ϵ is the Greek letter **epsilon**; by mathematical tradition, it's used in situations like this. But you could use another symbol if you wanted.) That is, if someone challenges me with ϵ, I have to show that I can find a large enough k so that

$$\epsilon > |a_k - 0| = |a_k|.$$

Since $\sum_{k=1}^{\infty} a_k$ converges to a sum S, the partial sums s_n must converge to S. Hence, I can choose k to be big enough so that s_k and s_{k-1} are within $\frac{1}{2} \cdot \epsilon$ of S. Then

$$|s_k - S| + |S - s_{k-1}| < \frac{1}{2} \cdot \epsilon + \frac{1}{2} \cdot \epsilon = \epsilon.$$

But by the Triangle Inequality,

$$|s_k - S| + |S - s_{k-1}| \geq |s_k - S + S - s_{k-1}| = |s_k - s_{k-1}| = |a_k|.$$

Therefore,

$$\epsilon > |s_k - S| + |S - s_{k-1}| \geq |a_k|.$$

As I noted earlier, this means that $\lim_{k\to\infty} a_k = 0$. □

You will more often use the following statement, which is logically equivalent to the statement I proved. I'll also call it the Zero Limit Test.

Corollary. (**Zero Limit Test**) If $\lim_{k\to\infty} a_k \neq 0$, then $\sum_{k=1}^{\infty} a_k$ diverges.

Example. Apply the Zero Limit Test to $\sum_{k=1}^{\infty} \frac{5k}{7k+3}$.

$$\lim_{k\to\infty} \frac{5k}{7k+3} = \frac{5}{7} \neq 0.$$

Hence, the series diverges, by the Zero Limit Test. □

Example. Apply the Zero Limit Test to $\sum_{k=1}^{\infty} \cos k$.

$$\lim_{k\to\infty} \cos k \quad \text{does not exist.}$$

Hence, the series diverges, by the Zero Limit Test. □

Remark. A standard mistake is to use the Zero Limit Test backward. It is *not* true that if $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=1}^{\infty} a_n$ converges. Counterexample: The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. □