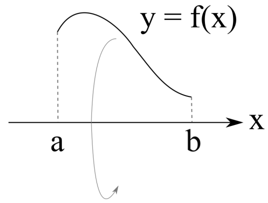
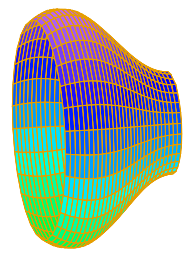
Surfaces of Revolution

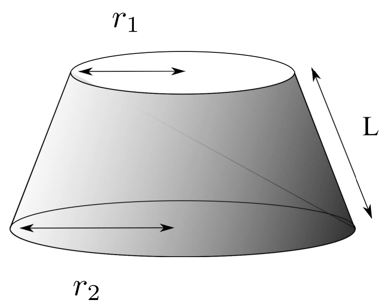
Suppose a curve $y = f(x)$ for $a \leq x \leq b$ is revolved about the x-axis. The curve sweeps out a surface in 3 dimensions.





To find the area of such a surface, I'll break the surface up into small pieces whose areas I can find. I will use pieces which are parts of cones.

A **frustum** of a cone is obtaining by cutting the cone with a plane parallel to its base.



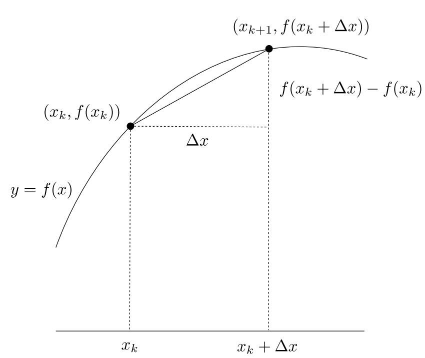
The area of a frustum of a cone is the product of the side diagonal and the average of the lengths of the top and bottom circles:

$$A = \frac{2\pi r_1 + 2\pi r_2}{2} \cdot L = 2\pi \frac{r_1 + r_2}{2} L.$$

Now consider a curve $y = f(x)$ for $a \leq x \leq b$ is revolved about the x-axis. Divide the interval up into n subintervals — for simplicity, we'll take them to have equal lengths $\Delta x = \dfrac{b-a}{n}$. The interval $a \leq x \leq b$ is partitioned into n subintervals:

$$a = x_0,\ x_1,\ x_2,\ \ldots x_{n-1} = b.$$

Consider a subinterval from x_k to $x_{k+1} = x_k + \Delta x$. On this subinterval, approximate the curve with the segment from $(x_k, f(x_k))$ to $(x_{k+1}, f(x_{k+1}))$.



The length of the curve can be approximated by the length of the segment, which is

$$\sqrt{\Delta x^2 + (f(x_k + \Delta x) - f(x_k))^2} = \sqrt{1 + \left(\frac{f(x_k + \Delta x) - f(x_k)}{\Delta x}\right)^2}\,\Delta x.$$

Assuming that f is differentiable on the interval, I can apply the Mean Value Theorem to find a number c_k such that $x_k < c_k < x_k + \Delta x$, and

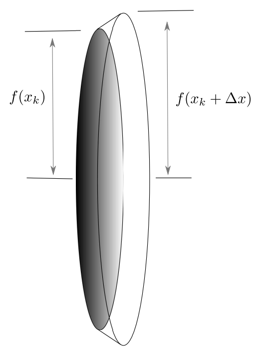
$$\frac{f(x_k + \Delta x) - f(x_k)}{\Delta x} = f'(c_k).$$

The length of the segment becomes

$$\sqrt{1 + f'(c_k)^2}\Delta x.$$

As the segment is revolved about the x-axis, it sweeps out a frustum of a cone (lying on its side) which approximates the area of the corresponding part of the surface of revolution. The side diagonal is the length of the segment, which we've seen is $\sqrt{1 + f'(c_k)^2}\Delta x$.

The two circles which bound the frustum have radii $f(x_k)$ and $f(x_k + \Delta x)$.



The area of the frustum is

$$2\pi \frac{f(x_k) + f(x_k + \Delta x)}{2} \sqrt{1 + f'(c_k)^2}\Delta x.$$

If I sum over all the subintervals, I get an approximation to the area of the surface:

$$\sum_{k=0}^{n-1} 2\pi \frac{f(x_k) + f(x_k + \Delta x)}{2} \sqrt{1 + f'(c_k)^2}\Delta x.$$

The area of the surface should be the limit of this sum as the number of subintervals goes to ∞ (i.e. as the length of the subintervals goes to 0):

$$A = \lim_{n\to\infty} \sum_{k=0}^{n-1} 2\pi \frac{f(x_k) + f(x_k + \Delta x)}{2} \sqrt{1 + f'(c_k)^2}\Delta x.$$

This is not a Riemann sum as such, since it depends on two values x_k and c_k. We may reason informally as follows.

As $\Delta x \to 0$, I have $x_k + \Delta x \to x_k$. So

$$\frac{f(x_k) + f(x_k + \Delta x)}{2} \to \frac{2f(x_k)}{2} = f(x_k).$$

In addition, c_k is in the interval from x_k to $x_k + \Delta x$, so $x_k \approx c_k$.

So we can identify the limit of the sum as the following integral:

$$A = \lim_{n\to\infty} \sum_{k=0}^{n-1} 2\pi \frac{f(x_k) + f(x_k + \Delta x)}{2} \sqrt{1 + f'(c_k)^2}\Delta x = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx.$$

A careful justification would require the use of more advanced results such as Bliss's Theorem.

I will restate the result in the following form, which makes the analogous formulas to follow easier to understand. For a curve $y = f(x)$,

$$A = \int_a^b 2\pi R \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Here R is the distance from the curve to the axis of revolution. Before explaining R, here are the other two formulas.

For a curve $x = g(y)$,

$$A = \int_a^b 2\pi R \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

For a curve given parametrically by $x = f(t)$, $y = g(t)$,

$$A = \int_a^b 2\pi R \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

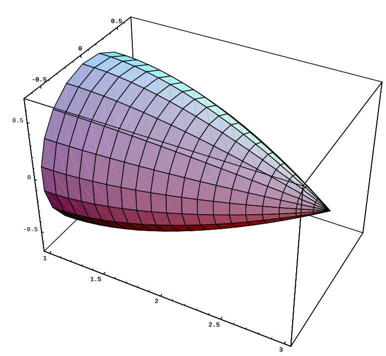
You can see that they are analogous to the arc length formulas, with the addition of a $2\pi R$ term. Informally, the formula multiplies the length of the curve by the distance it travels in a circle to get the area.

R is the distance from the curve to the axis, so

$$R = \begin{cases} |y| & \text{if the axis is the } x\text{-axis} \\ |x| & \text{if the axis is the } y\text{-axis} \end{cases}$$

If you know y or x is positive (as in many examples), you may drop the absolute values. You substitute for y or x as appropriate to get the correct variables for the integral.

Example. Find the area of the surface generated by revolving $y = x^{1/2} - \frac{1}{3}x^{3/2}$, $1 \le x \le 3$, about the x-axis.



Since the curve is being revolved about the x-axis, $R = y = x^{1/2} - \frac{1}{3}x^{3/2}$.

$$\begin{aligned} 1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \left(\frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2}\right)^2 \\ &= 1 + \frac{1}{4}x^{-1} - \frac{1}{2} + \frac{1}{4}x \\ &= \frac{1}{4}x^{-1} + \frac{1}{2} + \frac{1}{4}x \\ &= \left(\frac{1}{2}x^{-1/2} + \frac{1}{2}x^{1/2}\right)^2 \end{aligned}$$

Hence,

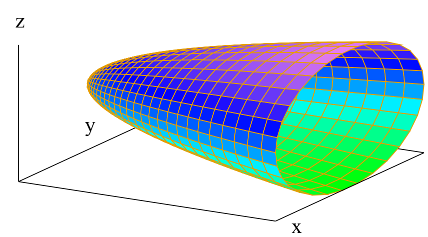
$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{1}{2}x^{-1/2} + \frac{1}{2}x^{1/2}.$$

The area is

$$S = \int_1^3 2\pi \left(\sqrt{x} - \frac{1}{3}x^{3/2}\right) \cdot \frac{1}{2}\left(x^{-1/2} + x^{1/2}\right)\, dx = \pi \int_1^3 \left(1 + \frac{2}{3}x - \frac{1}{3}x^2\right) dx =$$

$$\pi \left[x + \frac{1}{3}x^2 - \frac{1}{9}x^3\right]_1^3 = \frac{16\pi}{9} = 5.58505\ldots. \quad \square$$

Example. Find the area of the surface generated by revolving $x = \dfrac{1}{2}y^2 + 3$ for $0 \le y \le 1$ about the x-axis.



Since the curve is being revolved about the x-axis, $R = y$.

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + y^2.$$

Hence,

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1+y^2}.$$

The area is

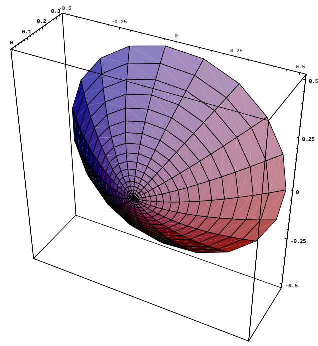
$$\int_0^1 2\pi y\sqrt{1+y^2}\, dy = 2\pi \int_?^? y\sqrt{u} \cdot \frac{du}{2y} = \pi \int_?^? \sqrt{u}\, du =$$

$$\left[u = 1 + y^2, \quad du = 2y\, dy, \quad dy = \frac{du}{2y}\right]$$

$$\frac{2\pi}{3}\left[u^{3/2}\right]_?^? = \frac{2\pi}{3}\left[(1+y^2)^{3/2}\right]_0^1 = \frac{2\pi}{3}(2^{3/2} - 1) = 3.82944\ldots. \quad \square$$

Example. Find the area of the surface generated by revolving the following curve about the y-axis:

$$x = \frac{1}{2}t^2, \quad y = \frac{1}{3}t^3, \quad 0 \le t \le 1.$$



The curve is revolved about the y-axis, so $R = x = \dfrac{1}{2}t^2$. This is positive, so no absolute values are needed.

$$\frac{dx}{dt} = t, \quad \frac{dy}{dt} = t^2.$$

$$\begin{aligned}\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= t^2 + t^4 \\ &= t^2(1 + t^2)\end{aligned}$$

Hence,

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = t\sqrt{t^2 + 1}.$$

The area is

$$S = \int_0^1 2\pi \cdot \frac{1}{2}t^2 \cdot t\sqrt{t^2 + 1}\, dt = \frac{\pi}{2}\left[\frac{2}{5}(t^2 + 1)^{5/2} - \frac{2}{3}(t^2 + 1)^{3/2}\right]_0^1 = \left(\frac{2}{15} + \frac{2\sqrt{2}}{15}\right)\pi = 0.32189\ldots.$$

Here's the work for the antiderivative:

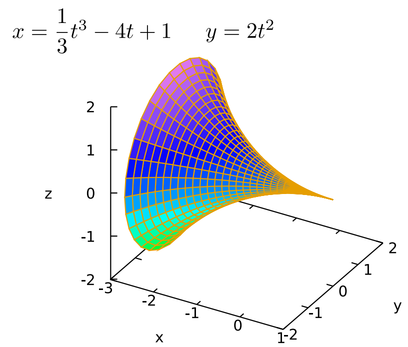
$$\int t^2 \cdot t\sqrt{t^2 + 1}\, dt = \frac{1}{2}\int (u - 1)\sqrt{u}\, du = \frac{1}{2}\int (u^{3/2} - u^{1/2})\, du = \frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + c =$$

$$\left[u = t^2 + 1, \quad t^2 = u - 1, \quad du = 2t\, dt, \quad dt = \frac{du}{2t}\right]$$

$$\frac{1}{2}\left(\frac{2}{5}(t^2 + 1)^{5/2} - \frac{2}{3}(t^2 + 1)^{3/2}\right) + c. \quad \square$$

Example. Find the area of the surface generated by revolving the following curve about the x-axis:

$$x = \frac{1}{3}t^3 - 4t + 1, \quad y = 2t^2, \quad 0 \le t \le 1.$$



The curve is revolved about the x-axis, so $R = y = 2t^2$. This is positive, so no absolute values are needed.

$$\frac{dx}{dt} = t^2 - 4, \quad \frac{dy}{dt} = 4t.$$

$$\begin{aligned}\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= (t^2 - 4)^2 + (4t)^2 \\ &= t^4 - 8t^2 + 16 + 16t^2 \\ &= t^4 + 8t^2 + 16 \\ &= (t^2 + 4)^2\end{aligned}$$

Hence,

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = t^2 + 4.$$

The area is

$$\int_0^1 2\pi \cdot 2t^2(t^2 + 4)\, dt = 4\pi \int_0^1 (t^4 + 4t^2)\, dt = 4\pi \left[\frac{1}{5}t^5 + \frac{4}{3}t^3\right]_0^1 = \frac{92\pi}{15} = 19.26843\ldots. \quad \square$$
