Taylor Series

The **Taylor series** for $f(x)$ at $x = c$ is

$$f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

(By convention, $f^{(0)} = f$.) When $c = 0$, the series is called a **Maclaurin series**.

You can construct the series on the right provided that f is infinitely differentiable on an interval containing c. You already know how to determine the interval of convergence of the series. However, the fact that the series converges at x does not imply that the series converges to $f(x)$.

As an example, consider the function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

It is infinitely differentiable everywhere. In particular, all the derivatives of f at 0 vanish, and the Maclaurin series for f is identically 0.

Hence, the Maclaurin series for f converges for all x, but only converges to $f(x)$ at $x = 0$.

The following result ([1], page 418) gives a sufficient condition for the Taylor series of a function to converge to the function:

Theorem. Let $f(x)$ be infinitely differentiable on $a \le x \le b$, and let $a \le c \le b$. Suppose there is a constant M such that $|f^{(n)}(x)| \le M$ for all $n \ge 1$, and for all x in $N \cap [a, b]$, where N is a neighborhood of c. Then for all $x \in N \cap [a, b]$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

In other words, under reasonable conditions:

1. You can construct a Taylor series by computing the derivatives of f.

2. The series will converge to f on an interval around the expansion point. (You can find the interval of convergence as usual.)

It's tedious to have to compute lots of derivatives, and in many cases you can derive a series from another known series. Here are the series expansions for several important functions:

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n = 1 + u + u^2 + \cdots + u^n + \cdots \qquad -1 < u < 1$$

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} = 1 + u + \frac{u^2}{2!} + \cdots + \frac{u^n}{n!} + \cdots \qquad -\infty < u < +\infty$$

$$\cos u = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n}}{(2n)!} = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \cdots + (-1)^n \frac{u^{2n}}{(2n)!} + \cdots \qquad -\infty < u < +\infty$$

$$\sin u = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{(2n+1)!} = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots + (-1)^n \frac{u^{2n+1}}{(2n+1)!} + \cdots \qquad -\infty < u < +\infty$$

$$\ln(1+u) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{u^n}{n} = u - \frac{u^2}{2} + \frac{u^3}{3} - \cdots + (-1)^{n+1} \frac{u^n}{n} + \cdots \qquad -1 < u \le 1$$

$$(1+u)^a = 1 + \sum_{n=1}^{\infty} \frac{a(a-1)\cdots(a-n+1)}{n!} u^n \qquad -1 < u < 1$$

Example. Find the Taylor series for $\dfrac{1}{x+3}$ at $a = 2$. What is its interval of convergence?

I want things to come out in powers of $x - 2$, so I'll write the function in terms of $x - 2$:

$$\begin{aligned} \frac{1}{x+3} &= \frac{1}{\ldots + (x-2)} && \text{(Make the } x-2 \text{ first)} \\ &= \frac{1}{5 + (x-2)} && \text{(I need 5, because } 5 - 2 = 3\text{)} \end{aligned}$$

I'll use the series for $\dfrac{1}{1-u}$. To do this, I need $1 - u$ on the bottom. I make a "1" by factoring 5 out of the terms on the bottom, then I make a "−" by writing the "+" as "−(−)":

$$\frac{1}{5+(x-2)} = \frac{1}{5} \cdot \frac{1}{1 + \dfrac{x-2}{5}} = \frac{1}{5} \cdot \frac{1}{1 - \left(-\dfrac{x-2}{5}\right)}.$$

Let $u = -\dfrac{x-2}{5}$ in the series for $\dfrac{1}{1-u}$. Then

$$\frac{1}{1 - \left(-\dfrac{x-2}{5}\right)} = 1 - \frac{x-2}{5} + \left(\frac{x-2}{5}\right)^2 - \left(\frac{x-2}{5}\right)^3 + \cdots.$$

Hence,

$$\frac{1}{x+3} = \frac{1}{5} \cdot \left[1 - \frac{x-2}{5} + \left(\frac{x-2}{5}\right)^2 - \left(\frac{x-2}{5}\right)^3 + \cdots\right].$$

The u-series converges for $-1 < u < 1$, so the x-series converges for $-1 < -\dfrac{x-2}{5} < 1$, or $-3 < x < 7$.

□

Example. Find the Taylor series for $\dfrac{1}{7-x}$ at $a = -3$. What is its interval of convergence?

Since I'm expanding at $a = -3$, I need powers of $x + 3$:

$$\begin{aligned} \frac{1}{7-x} &= \frac{1}{10 - (x+3)} \\ &= \frac{1}{10} \frac{1}{1 - \dfrac{1}{10}(x+3)} \end{aligned}$$

I let $u = \dfrac{1}{10}(x+3)$ in the series for $\dfrac{1}{1-u}$:

$$\frac{1}{10} \frac{1}{1 - \dfrac{1}{10}(x+3)} = \frac{1}{10}\left(1 + \frac{1}{10}(x+3) + \frac{1}{10^2}(x+3)^2 + \frac{1}{10^3}(x+3)^3 + \cdots\right).$$

In summation form, this is $\dfrac{1}{10}\sum_{n=0}^{\infty} \dfrac{1}{10^n}(x+3)^n$.

Find the interval of convergence:

$$\begin{aligned} -1 &< u < 1 \\ -1 &< \frac{1}{10}(x+3) < 1 \\ -10 &< x+3 < 10 \\ -13 &< x < 7 \end{aligned} \quad \square$$

Example. Find the Taylor series at $c = 1$ for e^{5x}.

I need powers of $x - 1$.

$$e^{5x} = e^{5(x-1)+5} = e^{5(x-1)} \cdot e^5 = e^5 \left(1 + 5(x-1) + \frac{5^2(x-1)^2}{2!} + \frac{5^3(x-1)^3}{3!} + \cdots\right).$$

To get this, I let $u = 5(x-1)$ in the series for e^u.
For the interval of convergence:

$$\begin{aligned} -\infty &< u < \infty \\ -\infty &< 5(x-1) < \infty \\ -\infty &< x-1 < \infty \\ -\infty &< x < \infty \end{aligned} \quad \square$$

Example. Find the Taylor series for $\sin x$ at $c = \dfrac{\pi}{2}$.

I need powers of $x - \dfrac{\pi}{2}$, so

$$\sin x = \sin\left[\left(x - \frac{\pi}{2}\right) + \frac{\pi}{2}\right].$$

Next, I'll use the angle addition formula for sine:

$$\sin(a+b) = \sin a \cos b + \sin b \cos a.$$

I set $a = x - \dfrac{\pi}{2}$ and $b = \dfrac{\pi}{2}$. Since $\cos\dfrac{\pi}{2} = 0$ and $\sin\dfrac{\pi}{2} = 1$, I get

$$\sin\left[\left(x - \frac{\pi}{2}\right) + \frac{\pi}{2}\right] = \cos\left(x - \frac{\pi}{2}\right) = 1 - \frac{1}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 - \frac{1}{6!}\left(x - \frac{\pi}{2}\right)^6 + \cdots. \quad \square$$

Example. Find the Taylor series for $\ln x$ at $a = 1$. What is its interval of convergence?

Use

$$\ln(1+u) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{u^n}{n} = u - \frac{u^2}{2} + \frac{u^3}{3} - \cdots + (-1)^{n+1}\frac{u^n}{n} + \cdots.$$

I'm expanding at $a = 1$, so I want the result to come out in powers of $x - 1$. This is easy — just set $u = x - 1$:

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \cdots + (-1)^{n+1}\frac{1}{n}(x-1)^n + \cdots.$$

The u-series converges for $-1 < u \le 1$, so the x-series converges for $-1 < x - 1 \le 1$, or $0 < x \le 2$. $\square$

Example. The quantity $\left(1 - \dfrac{v^2}{c^2}\right)^{-1/2}$ occurs in special relativity. (v is the velocity of an object, and c is the speed of light.) Approximate $\left(1 - \dfrac{v^2}{c^2}\right)^{-1/2}$ using the first two nonzero terms of the binomial series.

$$(1+u)^a = 1 + au + \frac{a(a-1)}{2!}u^2 + \cdots,$$

So for $a = -\dfrac{1}{2}$,

$$(1+u)^{-1/2} = 1 - \frac{1}{2}u + \frac{3}{8}u^2 - \cdots.$$

Take $u = -\dfrac{v^2}{c^2}$:

$$\left(1 - \frac{v^2}{c^2}\right)^{-1/2} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots \approx 1 + \frac{1}{2}\frac{v^2}{c^2}.$$

The approximation is good as long as v is small compared to c. $\square$

Example. Find the Taylor series for $\dfrac{x}{2+x}$ at $a = -1$.

Since I'm expanding at $a = -1$, the answer must come out in terms of powers of $x + 1$.

Start with the function you're trying to expand. To get $x + 1$'s in the answer, write the given function in terms of $x + 1$:

$$\frac{x}{2+x} = \frac{(x+1)-1}{1+(x+1)}.$$

(Notice that the work has to be legal algebra.)

I'll break up the fraction and do the pieces separately.

$$\frac{(x+1)-1}{1+(x+1)} = \frac{x+1}{1+(x+1)} - \frac{1}{1+(x+1)}.$$

I want to "match" each piece against the standard series $\dfrac{1}{1-u}$. Here's the first piece:

$$\frac{x+1}{1+(x+1)} = (x+1)\frac{1}{1-[-(x+1)]}.$$

Expand $\dfrac{1}{1-[-(x+1)]}$ by setting $u = -(x+1)$ in $\dfrac{1}{1-u}$:

$$(x+1)\frac{1}{1-[-(x+1)]} = (x+1)\cdot\left(1-(x+1)+(x+1)^2-(x+1)^3+\cdots\right) = (x+1)-(x+1)^2+(x+1)^3-\cdots.$$

Here's the second piece:

$$\frac{1}{1+(x+1)} = \frac{1}{1-[-(x+1)]} = 1-(x+1)+(x+1)^2-(x+1)^3+\cdots.$$

Put the two pieces together:

$$\left[(x+1)-(x+1)^2+(x+1)^3-\cdots\right] - \left[1-(x+1)+(x+1)^2-(x+1)^3+\cdots\right] =$$

$$\begin{array}{ccccccccc} & & (x+1) & - & (x+1)^2 & + & (x+1)^3 & - & \cdots \\ -1 & + & (x+1) & - & (x+1)^2 & + & (x+1)^3 & - & \cdots \end{array} =$$

$$-1 + 2(x+1) - 2(x+1)^2 + 2(x+1)^3 - \cdots.$$

That is,

$$\frac{x}{2+x} = -1 + 2(x+1) - 2(x+1)^2 + 2(x+1)^3 - \cdots. \quad \square$$

Example. What is the Maclaurin series for $f(x) = 7x^2 - 3x + 13$? What is the Taylor series for $f(x) = 7x^2 - 3x + 13$ at $a = -1$?

The Maclaurin series for a polynomial is the polynomial: $f(x) = 7x^2 - 3x + 13$.
To obtain the Taylor expansion at $a = -1$, write the function in terms of $x + 1$:

$$7x^2 - 3x + 13 = 7(x+1)^2 - 17x + 6 = 7(x+1)^2 - 17(x+1) + 23. \quad \square$$

Example. Find $f^{(100)}(0)$ for $f(x) = \dfrac{1}{3-x}$.

The series for $\dfrac{1}{3-x}$ at $c = 0$ is

$$\frac{1}{3-x} = \frac{1}{3} \cdot \frac{1}{1 - \dfrac{x}{3}} = \frac{1}{3} \cdot \left(1 + \frac{x}{3} + \frac{x^2}{3^2} + \cdots + \frac{x^n}{3^n} + \cdots\right) =$$

$$\frac{1}{3} + \frac{x}{3^2} + \frac{x^2}{3^3} + \cdots + \frac{x^n}{3^{n+1}} + \cdots.$$

The 100^{th} degree term is $\dfrac{x^{100}}{3^{101}}$. On the other hand, Taylor's formula says that the 100^{th} degree term is $\dfrac{f^{(100)}(0)}{100!}x^{100}$. Equating the coefficients, I get

$$\frac{1}{3^{101}} = \frac{f^{(100)}(0)}{100!}$$

$$f^{(100)}(0) = \frac{100!}{3^{101}} \quad \square$$

While you can often use known series to find Taylor series, it's sometimes necessary to find a series using Taylor's formula. (In fact, that's where the "known series" come from.)

Example. Find the first four nonzero terms and the general term of the Taylor series for $f(x) = e^x$ at $a = 0$ and at $a = 1$ by computing the derivatives of f.

$$f(x) = e^x, \quad f'(x) = e^x, \quad \text{and in general} \quad f^{(n)}(x) = e^x.$$

For $a = 0$, $f^{(n)}(0) = e^0 = 1$ for all n. The Taylor series at $a = 0$ is

$$f(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots.$$

For $a = 1$, $f^{(n)}(1) = e^1 = e$ for all n. The Taylor series at $a = 1$ is

$$f(x) = e + e(x-1) + \frac{e}{2!}(x-1)^2 + \frac{3}{3!}(x-1)^3 + \cdots + \frac{1}{n!}(x-1)^n + \cdots. \quad \square$$

If you truncate the series expanded at c after the n^{th}-degree term, what's left is the n^{th}**-degree Taylor polynomial** $p_n(x;c)$. For example, the third degree polynomial of e^x at $a = 0$ is

$$p_3(x;0) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3.$$

Note that the "n" here refers to the largest *power of* x, not the number of terms. For example, the Taylor series for $\dfrac{1}{1-x^2}$ at $a = 0$ is

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + \cdots + x^{2n} + \cdots.$$

The 2^{nd} degree Taylor polynomial and the 3^{rd} degree Taylor polynomial are equal:

$$p_2(x;0) = p_3(x;0) = 1 + x^2. \quad \square$$

Example. Find the 3^{rd} degree Taylor polynomial for $f(x) = \tan x$ at $x = \dfrac{\pi}{4}$.

$$f(x) = \tan x, \quad f'(x) = (\sec x)^2, \quad f''(x) = 2(\sec x)^2 \tan x, \quad f'''(x) = 2(\sec x)^4 + 4(\sec x)^2(\tan x)^2.$$

Thus,

$$f\left(\frac{\pi}{4}\right) = 1, \quad f'\left(\frac{\pi}{4}\right) = 2, \quad f''\left(\frac{\pi}{4}\right) = 4, \quad f'''\left(\frac{\pi}{4}\right) = 16.$$

The 3^{rd} degree Taylor polynomial is

$$p_3\left(x;\frac{\pi}{4}\right) = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3. \quad \square$$

Example. Suppose

$$f(4) = 7, \quad f'(4) = -3, \quad f''(4) = 4, \quad f'''(4) = 12.$$

Use the 3^{rd} degree Taylor polynomial for f at $c = 4$ to approximate $f(4.2)$.

I have

$$p_3(x;4) = 7 - 3(x-4) + \frac{4}{2!}(x-4)^2 + \frac{12}{3!}(x-4)^3 = 7 - 3(x-4) + 2(x-4)^2 + 2(x-4)^3.$$

Plug $x = 4.2$ in:

$$f(4.2) \approx 7 - 3(4.2-4) + 2(4.2-4)^2 + 2(4.2-4)^3 = 6.496. \quad \square$$

It's also possible to construct power series by integrating or differentiating other power series. *A power series may be integrated or differentiated term-by-term in the interior of its interval of convergence.* (You will need to check convergence at the endpoints separately.)

Example. (a) Find the Taylor series at $c = 0$ for $\dfrac{1}{8+x}$.

(b) Find the Taylor series at $c = 0$ for $\dfrac{1}{(8+x)^2}$.

(a)

$$\frac{1}{8+x} = \frac{1}{8}\frac{1}{1+\dfrac{x}{8}} = \frac{1}{8}\frac{1}{1-\left(-\dfrac{x}{8}\right)} =$$

$$\frac{1}{8}\left(1 - \frac{x}{8} + \frac{x^2}{64} - \frac{x^3}{512} + \frac{x^4}{4096} - \cdots\right). \quad \square$$

(b) Notice that

$$\frac{d}{dx}\frac{1}{8+x} = -\frac{1}{(8+x)^2}.$$

Hence,

$$\frac{1}{(8+x)^2} = -\frac{d}{dx}\frac{1}{8+x} = -\frac{d}{dx}\frac{1}{8}\left(1 - \frac{x}{8} + \frac{x^2}{64} - \frac{x^3}{512} + \frac{x^4}{4096} - \cdots\right) =$$

$$-\frac{1}{8}\left(-\frac{1}{8} + \frac{x}{32} - \frac{3x^2}{512} + \frac{x^3}{1024} - \cdots\right). \quad \square$$

Example. (a) Find the Taylor series at $c = 0$ for $\dfrac{1}{1+x}$.

(b) Use the series in (a) to find the series for $\ln(1+u)$ expanded at $c = 0$.

(a) Put $u = -x$ in the series for $\dfrac{1}{1-u}$ to obtain

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots.$$

It converges for $-1 < x < 1$. $\square$

(b) Integrate the series in (a) from 0 to u:

$$\ln(1+u) = \int_0^u \left(1 - x + x^2 - x^3 + \cdots\right)\, dx = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \cdots.$$

This series will converge for $-1 < u < 1$. The left side blows up at $u = -1$. On the other hand, if $u = 1$,

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$$

The right side *does* converges (by the Alternating Series Test), so the $\ln(1+u)$ series converges for $-1 < u \leq 1$. $\square$

Example. Find the Taylor series for $\ln(5-x)$ at $a = 2$.

First, note that

$$\int_2^x \frac{1}{5-t}\,dt = [-\ln(5-t)]_2^x = -\ln(5-x) + \ln 3, \qquad \text{so} \qquad \ln(5-x) = \ln 3 - \int_2^x \frac{1}{5-t}\,dt.$$

I integrated from 2 to x because I want the expansion at $a = 2$.
Now find the series at $a = 2$ for $\dfrac{1}{5-t}$:

$$\frac{1}{5-t} = \frac{1}{3-(t-2)} = \frac{1}{3}\frac{1}{1-\dfrac{t-2}{3}} = \frac{1}{3}\sum_{n=0}^{\infty}\frac{(t-2)^n}{3^n}.$$

Plug this series back into the integral and integrate term-by-term:

$$\ln(5-x) = \ln 3 - \int_2^x \frac{1}{5-t}\,dt = \ln 3 - \frac{1}{3}\int_2^x \sum_{n=0}^{\infty}\frac{(t-2)^n}{3^n}\,dt = \ln 3 - \frac{1}{3}\sum_{n=0}^{\infty}\left[\frac{(t-2)^{n+1}}{3^n(n+1)}\right]_2^x =$$

$$\ln 3 - \frac{1}{3}\sum_{n=0}^{\infty}\frac{(x-2)^{n+1}}{3^n(n+1)} = \ln 3 - \sum_{n=0}^{\infty}\frac{(x-2)^{n+1}}{3^{n+1}(n+1)}. \quad \square$$

Example. (a) Construct the Taylor series at $c = 0$ for $\dfrac{1}{1+t^2}$.

(b) Use the series in (a) to construct the Taylor series at $c = 0$ for $\tan^{-1} x$.

(c) Use the series in (b) to obtain a series for π.

(a) I need powers of t, so

$$\frac{1}{1+t^2} = \frac{1}{1-(-t^2)} = 1 - t^2 + t^4 - t^6 + \cdots + (-1)^n t^{2n} + \cdots. \quad \square$$

(b) Note that

$$\int_0^x \frac{1}{1+t^2}\,dt = \left[\tan^{-1} t\right]_0^x = \tan^{-1} x.$$

Therefore,

$$\tan^{-1} x = \int_0^x \frac{1}{1+t^2}\,dt = \int_0^x \left(1 - t^2 + t^4 - t^6 + \cdots\right)\,dt =$$

$$\left[t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7 + \cdots\right]_0^x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots. \quad \square$$

(c) Plug $x = 1$ into the series in (b), using the fact that $\tan^{-1} 1 = \dfrac{\pi}{4}$:

$$\tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \quad \square$$

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots$$

Think of a Taylor series as a "replacement" for its function. For example, you can often use a Taylor series to compute a limit or an integral by replacing a function with its series.

Example. (a) Find the first 4 nonzero terms of the Taylor series at $c = 0$ for $\ln(1 + x^3)$.

(b) Use the series in (a) to guess the value of $\displaystyle \lim_{x\to 0} \frac{\ln(1 + x^3)}{x^3}$.

(a) Let $u = x^3$ in the series for $\ln(1 + u)$:

$$\ln(1 + x^3) = x^3 - \frac{1}{2}x^6 + \frac{1}{3}x^9 + \frac{1}{4}x^{12} - \cdots. \quad \square$$

(b) Plug the series from (a) into the limit:

$$\lim_{x\to 0} \frac{\ln(1 + x^3)}{x^3} = \lim_{x\to 0} \frac{1}{x^3}\left(x^3 - \frac{1}{2}x^6 + \frac{1}{3}x^9 + \frac{1}{4}x^{12} - \cdots\right) = \lim_{x\to 0}\left(1 - \frac{1}{2}x^3 + \frac{1}{3}x^6 + \frac{1}{4}x^9 - \cdots\right) = 1. \quad \square$$

Example. (a) Construct the Taylor series at $c = 0$ for $x^2e^{-x^2}$. (Write out at least the first 4 nonzero terms.)

(b) Use the first 3 terms of the series in (a) to approximate $\displaystyle \int_0^1 x^2e^{-x^2}\,dx$.

(c) Use the Alternating Series error estimate to estimate the error in (b).

(a) I set $u = -x^2$ in the series for e^u:

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots.$$

Multiply by x^2:

$$x^2e^{-x^2} = x^2 - x^4 + \frac{1}{2}x^6 - \frac{1}{6}x^8 + \frac{1}{24}x^{10} - \cdots. \quad \square$$

(b)

$$\int_0^1 x^2e^{-x^2}\,dx \approx \int_0^1 \left(x^2 - x^4 + \frac{1}{2}x^6\right)dx = \left[\frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{1}{14}x^7\right]_0^1 = \frac{43}{210} = 0.20476\ldots. \quad \square$$

(c) The Alternating Series error estimate says that the error is less than the next term. So I take the next term in the series in (a) and integrate:

$$\int_0^1 \frac{1}{6}x^8\,dx = \left[\frac{1}{54}x^9\right]_0^1 = \frac{1}{54}.$$

The error in the estimate in (b) is no greater than $\dfrac{1}{54} = 0.01851\ldots$. $\square$

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[1] Tom M. Apostol, *Mathematical Analysis*. Reading, Massachusetts: Addision-Wesley Publishing Company, Inc., 1957.