# Trigonometric Integrals

For trig integrals involving powers of sines and cosines, there are two important cases:

1. The integral contains an odd power of sine or cosine.

2. The integral contains only even powers of sines and cosines.

I will look at the odd power case first. It turns out that the same idea can be used to integrate some powers of secants and tangents, so I'll digress to do some examples of those as well.

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**Example.** Compute $\int (\sin 5x)^3 \left(1 + 4(\cos 5x)^2\right)\, dx$.

$$\int (\sin 5x)^3 \left(1 + 4(\cos 5x)^2\right)\, dx = \int (\sin 5x)^2 \left(1 + 4(\cos 5x)^2\right) \sin 5x\, dx =$$

$$\int \left(1 - (\cos 5x)^2\right) \left(1 + 4(\cos 5x)^2\right) \sin 5x\, dx =$$

$$\left[u = \cos 5x, \quad du = -5 \sin 5x\, dx, \quad dx = -\frac{du}{5 \sin 5x}\right]$$

$$-\frac{1}{5} \int (1 - u^2)(1 + 4u^2)\, du = -\frac{1}{5} \int (1 + 3u^2 - 4u^4)\, du = -\frac{1}{5}(u + u^3 - \frac{4}{5}u^5) + C =$$

$$-\frac{1}{5} \left(\cos 5x + (\cos 5x)^3 - \frac{4}{5}(\cos 5x)^5\right) + C = -\frac{1}{5} \cos 5x - \frac{1}{5}(\cos 5x)^3 + \frac{4}{25}(\cos 5x)^5 + C.$$

In this example, the key point was in the second line. I obtained an integral with lots of $\cos 5x$'s and a single $\sin 5x$. This allowed me to make the substitution $u = \cos 5x$, because the $\sin 5x$ was available to make $du$.

I got the $\sin 5x$ by "pulling it off" the odd power of $\sin 5x$. Then I converted the rest of the stuff to $\cos 5x$'s using the identity $(\sin \theta)^2 + (\cos \theta)^2 = 1$. This is the generic procedure when you have at least one odd power of sine or cosine. □

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**Example.** Compute $\int \left(5(\sin x)^{2/3} + 1\right) (\cos x)^3\, dx$.

$$\int \left(5(\sin x)^{2/3} + 1\right) (\cos x)^3\, dx = \int \left(5(\sin x)^{2/3} + 1\right) \left(1 - (\sin x)^2\right) (\cos x)\, dx =$$

$$\left[u = \sin x, \quad du = \cos x\, dx, \quad dx = \frac{du}{\cos x}\right]$$

$$\int \left(5u^{2/3} + 1\right) \left(1 - u^2\right)\, du = \int \left(5u^{2/3} + 1 - 5u^{8/3} - u^2\right)\, du = 3u^{5/3} + u - \frac{15}{11}u^{11/3} - \frac{1}{3}u^3 + C =$$

$$3(\sin x)^{5/3} + \sin x - \frac{15}{11}(\sin x)^{11/3} - \frac{1}{3}(\sin x)^3 + C. \quad □$$

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If you have an integral involving sines and cosines in which all the powers are *even*, the method I just described usually won't work. Instead, it is better to apply the following **double angle formulas**:

$$(\sin\theta)^2 = \frac{1}{2}(1-\cos 2\theta)$$
$$(\cos\theta)^2 = \frac{1}{2}(1+\cos 2\theta)$$

Any even power of $\sin x$ or $\cos x$ can be expressed as a power of $(\sin x)^2$ or $(\cos x)^2$. Use the identities above to substitute for $(\sin x)^2$ or $(\cos x)^2$, and multiply out the result. The net effect is to **reduce the powers** that occur in the integral, while at the same time increasing the arguments ($x \to 2x$).

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**Example.** Compute $\displaystyle\int (\cos 5x)^2(\sin 5x)^2\,dx$.

$$\int (\cos 5x)^2(\sin 5x)^2\,dx = \int \left(\frac{1}{2}(1+\cos 10x)\right)\left(\frac{1}{2}(1-\cos 10x)\right)\,dx = \frac{1}{4}\int \left(1-(\cos 10x)^2\right)\,dx =$$

$$\frac{1}{4}\int (\sin 10x)^2\,dx = \frac{1}{4}\int \frac{1}{2}(1-\cos 20x)\,dx = \frac{1}{8}(x - \frac{1}{20}\sin 20x) + C. \quad \square$$

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**Example.** Compute $\displaystyle\int (\sin 5x)^4\,dx$.

I'll use the double angle formula (twice):

$$\int (\sin 5x)^4\,dx = \int [(\sin 5x)^2]^2\,dx = \int \left[\frac{1}{2}(1-\cos 10x)\right]^2\,dx = \frac{1}{4}\int \left[1-2\cos 10x + (\cos 10x)^2\right]\,dx =$$

$$\frac{1}{4}\int \left[1-2\cos 10x + \frac{1}{2}(1+\cos 20x)\right]\,dx = \frac{1}{4}\left[x - \frac{1}{5}\sin 10x + \frac{1}{2}(x + \frac{1}{20}\sin 20x)\right] + c =$$

$$\frac{3}{8}x - \frac{1}{20}\sin 10x + \frac{1}{160}\sin 20x + c. \quad \square$$

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**Example.** Why would it be a bad idea to use the double angle formulas to compute $\displaystyle\int (\cos x)^3\,dx$?

Suppose I try to apply the double angle formula for cosine:

$$\int (\cos x)^3\,dx = \int (\cos x)^2\cos x\,dx = \int \frac{1}{2}(1+\cos 2x)\cos x\,dx.$$

The integral can be done in this form, but you either need to apply one of the angle addition formulas to $\cos 2x\cos x$ or use integration by parts. The problem is that *having trig functions with different arguments in the same integral makes the integral a bit harder to do.*

It would have been better to do the integral by using the "odd power" technique:

$$\int (\cos x)^3\,dx = \int (\cos x)^2\cos x\,dx = \int \left(1-(\sin x)^2\right)\cos x\,dx =$$

$$\left[u = \sin x, \quad du = \cos x\,dx, \quad dx = \frac{du}{\cos x}\right]$$

$$\int (1-u^2)\,du = u - \frac{1}{3}u^3 + C = \sin x - \frac{1}{3}(\sin x)^3 + C. \quad \square$$

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In some cases, you can use trig identities to do integrals involving sine and cosine. For example, the angle addition and subtraction formulas for cosine are

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$
$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$

Add the two equations — the $\sin a \sin b$ terms cancel — and divide by 2:

$$\cos(a+b) + \cos(a-b) = 2\cos a \cos b$$
$$\frac{1}{2}\left(\cos(a+b) + \cos(a-b)\right) = \cos a \cos b$$

The formula is

$$\cos a \cos b = \frac{1}{2}\left(\cos(a+b) + \cos(a-b)\right).$$

If instead you subtract the $\cos(a+b)$ equation from the $\cos(a-b)$ equation and divide by 2, you obtain

$$\sin a \sin b = \frac{1}{2}\left(\cos(a-b) - \cos(a+b)\right).$$

Likewise, the angle addition and subtraction formulas for sine are

$$\sin(a+b) = \sin a \cos b + \sin b \cos a$$
$$\sin(a-b) = \sin a \cos b - \sin b \cos a$$

Adding the equations and dividing by 2 gives

$$\sin a \cos b = \frac{1}{2}\left(\sin(a+b) + \sin(a-b)\right).$$

To summarize:

(a) $\cos a \cos b = \dfrac{1}{2}\left(\cos(a+b) + \cos(a-b)\right)$.

(b) $\sin a \sin b = \dfrac{1}{2}\left(\cos(a-b) - \cos(a+b)\right)$.

(c) $\sin a \cos b = \dfrac{1}{2}\left(\sin(a+b) + \sin(a-b)\right)$.

You can use these identities to integrate products of sines and cosines with different arguments. (Note that you can also do these integrals using integration by parts.)

**Example.** Compute $\displaystyle\int \sin 8x \cos 3x\,dx$.

Using the formula for $\sin a \cos b$ with $a = 8x$ and $b = 3x$, I get

$$\int \sin 8x \cos 3x\,dx = \frac{1}{2}\int (\sin 11x + \sin 5x)\ dx = \frac{1}{2}\left(-\frac{1}{11}\cos 11x - \frac{1}{5}\cos 5x\right) + c. \quad \square$$

Note: If you wind up with a "negative angle" in applying the identities, you can get rid of it using the identities

$$\sin(-\theta) = -\sin\theta \quad \text{and} \quad \cos(-\theta) = \cos\theta).$$

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To integrate some powers of secants and tangents, here are two useful approaches:

1. Use $(\sec\theta)^2 = 1 + (\tan\theta)^2$ to convert the integrand to something with lots of $\tan\theta$'s and a single $(\sec\theta)^2$. Then substitute $u = \tan\theta$.

2. Use $(\tan\theta)^2 = (\sec\theta)^2 - 1$ to convert the integrand to something with lots of $\sec\theta$'s and a single $\sec\theta\tan\theta$. Then substitute $u = \sec\theta$.

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**Example.** Compute $\displaystyle\int (\sec 3x)^4\,dx = \int (\sec 3x)^2(\sec 3x)^2\,dx$.

$$\int (\sec 3x)^4\,dx = \int (\sec 3x)^2(\sec 3x)^2\,dx = \int \left(1 + (\tan 3x)^2\right)(\sec 3x)^2\,dx =$$

$$\left[u = \tan 3x, \quad du = 3(\sec 3x)^2\,dx, \quad dx = \frac{du}{3(\sec 3x)^2}\right]$$

$$\frac{1}{3}\int (1 + u^2)\,du = \frac{1}{3}(u + \frac{1}{3}u^3) + C = \frac{1}{3}\tan 3x + \frac{1}{9}(\tan 3x)^3 + C.$$

In this example, I pulled off a $(\sec 3x)^2$, then converted the rest of the stuff to $\tan 3x$'s using $(\sec\theta)^2 = 1 + (\tan\theta)^2$. The $(\sec 3x)^2$ was exactly what I needed to make $du$ for the substitution $u = \tan 3x$.

Notice that the argument $3x$ did not play an important role in the problem. □

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**Example.** Compute $\displaystyle\int (\sec 6x)(\tan 6x)^3\,dx$.

$$\int (\sec 6x)(\tan 6x)^3\,dx = \int (\tan 6x)^2(\sec 6x\tan 6x\,dx) = \int \left((\sec 6x)^2 - 1\right)(\sec 6x\tan 6x\,dx) =$$

$$\left[u = \sec 6x, \quad du = 6\sec 6x\tan 6x\,dx, \quad dx = \frac{du}{6\sec 6x\tan 6x}\right]$$

$$\int (u^2 - 1)(\sec 6x\tan 6x)\cdot\frac{du}{6\sec 6x\tan 6x} = 6\int (u^2 - 1)\,du = 6\left(\frac{1}{3}u^3 - u\right) + C = 2(\sec 6x)^3 - 6\sec 6x + C.$$

In this example, I pulled off a $\sec 6x\tan 6x$, then converted the rest of the stuff to $\sec 6x$'s using $(\tan\theta)^2 = (\sec\theta)^2 - 1$. The $\sec 6x\tan 6x$ was exactly what I nneded to make $du$ for the substitution $u = \sec 6x$. □

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**Example.** Compute $\displaystyle\int (\tan\theta)^3\,d\theta$.

$$\int (\tan\theta)^3\,d\theta = \int (\tan\theta)^2\tan\theta\,d\theta = \int \left((\sec\theta)^2 - 1\right)\tan\theta\,d\theta =$$

$$\int (\sec\theta)^2\tan\theta\,d\theta - \int \tan\theta\,d\theta = \int \sec\theta(\sec\theta\tan\theta\,d\theta) - \int \frac{\sin\theta}{\cos\theta}\,d\theta.$$

I can do the first integral using $u = \sec\theta$, so $du = \sec\theta\tan\theta\,d\theta$ and $d\theta = \dfrac{du}{\sec\theta\tan\theta}$:

$$\int \sec\theta(\sec\theta\tan\theta\,d\theta) = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\sec\theta)^2 + C.$$

I can do the second integral using $w = \cos\theta$, so $dw = -\sin\theta\,d\theta$ and $d\theta = \dfrac{dw}{-\sin\theta}$:

$$\int \frac{\sin\theta}{\cos\theta}\,d\theta = \int \frac{1}{w}\,dw = \ln|w| + C = \ln|\cos\theta| + C.$$

Therefore,

$$\int (\tan\theta)^3\,d\theta = \frac{1}{2}(\sec\theta)^2 + \ln|\cos\theta| + C. \quad \square$$

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**Example.** Compute $\displaystyle\int \frac{1}{(\sec x)^2\tan x}\,dx$.

In this problem, I'll use the identity

$$(\sec x)^2 - (\tan x)^2 = 1.$$

Applying this to the top of the fraction, I get

$$\int \frac{1}{(\sec x)^2\tan x}\,dx = \int \frac{(\sec x)^2 - (\tan x)^2}{(\sec x)^2\tan x}\,dx = \int \left(\frac{(\sec x)^2}{(\sec x)^2\tan x} - \frac{(\tan x)^2}{(\sec x)^2\tan x}\right)dx =$$

$$\int \left(\frac{1}{\tan x} - \frac{\tan x}{(\sec x)^2}\right)dx = \int \cot x\,dx - \int \frac{\sin x}{\cos x}\cdot(\cos x)^2\,dx = \ln|\sin x| - \int \sin x\cos x\,dx =$$

$$\ln|\sin x| - \frac{1}{2}(\sin x)^2 + C.$$

I used the formula

$$\int \cot x\,dx = \ln|\sin x| + C.$$

If you didn't know this, you could derive it by writing $\cot x = \dfrac{\cos x}{\sin x}$. Then substitute $u = \sin x$.

I used $u = \sin x$ to do $\displaystyle\int \sin x\cos x\,dx$. $\square$

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**Example.** Compute $\displaystyle\int \sec x\,dx$.

This integral uses a trick:

$$\int \sec x\,dx = \int \sec x\cdot\frac{\sec x + \tan x}{\sec x + \tan x}\,dx = \int \frac{(\sec x)^2 + \sec x\tan x}{\sec x + \tan x}\,dx =$$

$$\left[u = \sec x + \tan x, \quad du = \left(\sec x\tan x + (\sec x)^2\right)\,dx, \quad dx = \frac{du}{\sec x\tan x + (\sec x)^2}\right]$$

$$\int \frac{(\sec x)^2 + \sec x\tan x}{u}\cdot\frac{du}{\sec x\tan x + (\sec x)^2} = \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C. \quad \square$$

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**Example.** Compute $\int (\sec x)^3\,dx$.

I can compute $\int (\sec x)^3\,dx$ using parts:

$$\int (\sec x)^3\,dx = \int (\sec x)(\sec x)^2\,dx = \sec x \tan x - \int \sec x (\tan x)^2\,dx =$$

| | $\dfrac{d}{dx}$ | | $\int dx$ |
|---|---|---|---|
| $+$ | $\sec x$ | $\searrow$ | $(\sec x)^2$ |
| $-$ | $\sec x \tan x$ | $\rightarrow$ | $\tan x$ |

$$\sec x \tan x - \int \sec x \left((\sec x)^2 - 1\right)\,dx = \sec x \tan x - \int \left((\sec x)^3 - \sec x\right)\,dx =$$

$$\sec x \tan x + \int \sec x\,dx - \int (\sec x)^3\,dx = \sec x \tan x + \ln|\sec x + \tan x| - \int (\sec x)^3\,dx.$$

Thus,

$$\int (\sec x)^3\,dx = \sec x \tan x + \ln|\sec x + \tan x| - \int (\sec x)^3\,dx$$

$$2\int (\sec x)^3\,dx = \sec x \tan x + \ln|\sec x + \tan x|$$

$$\int (\sec x)^3\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$$

This integral also comes up a lot, so you should make a note of it. □

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**Remark.** Using the methods of the last two examples, you can show:

$$\int \csc x\,dx = -\ln|\csc x + \cot x| + C.$$

$$\int (\csc x)^3\,dx = -\frac{1}{2}\csc x \cot x - \frac{1}{2}\ln|\csc x + \cot x| + C.$$

In general, integrals involving powers of cosecant and cotangent use the same ideas as integrals involving powers of secant and tangent. □

**Example.** Compute $\int (\cot 5x)^2\,dx$.

Remember the trig identity

$$(\csc x)^2 = 1 + (\cot x)^2.$$

So

$$\int (\cot 5x)^2\,dx = \int \left((\csc 5x)^2 - 1\right)\,dx = -\frac{1}{5}\cot 5x - x + C. \quad □$$

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The examples show that certain patterns that arise in trig integrals are good, in the sense that they allow you to do a substitution which makes the integral easy. Here is a review of some of the "good patterns":

(a) Lots of $\cos x$'s and a single $\sin x$.

(b) Lots of $\sin x$'s and a single $\cos x$.

(c) Lots of $\tan x$'s and a single $(\sec x)^2$.

(d) Lots of $\sec x$'s and a single $\sec x \tan x$.

(e) Lots of $\cot x$'s and a single $(\csc x)^2$.

(f) Lots of $\csc x$'s and a single $\csc x \cot x$.

You should aim for these patterns whenever possible.

Finally, I'll note that you can sometimes use integration by parts to obtain **recursion formulas** which reduce the integral of a power of a trig function to the integral of a smaller power.

**Example.** Derive a recursion formula for $\int (\sin x)^n \, dx$ for $n \geq 2$.

Integrate $\int (\sin x)^n \, dx$ by parts:

| | $\dfrac{d}{dx}$ | $\int dx$ |
|---|---|---|
| $+$ | $(\sin x)^{n-1}$ | $\sin x$ |
| $-$ | $(n-1)(\sin x)^{n-2}\cos x$ | $-\cos x$ |

$$\int (\sin x)^n \, dx = -(\sin x)^{n-1}\cos x + (n-1)\int (\sin x)^{n-2}(\cos x)^2 \, dx =$$

$$-(\sin x)^{n-1}\cos x + (n-1)\int (\sin x)^{n-2}[1-(\sin x)^2] \, dx =$$

$$-(\sin x)^{n-1}\cos x + (n-1)\int (\sin x)^{n-2} \, dx - (n-1)\int (\sin x)^n \, dx.$$

After integrating by parts, I used the identity $(\cos x)^2 = 1 - (\sin x)^2$. I multiplied out the terms in the integral, then broke the integral up into two integrals.

Next, add $(n-1)\int (\sin x)^n \, dx$ to both sides of the equation:

$$n\int (\sin x)^n \, dx = -(\sin x)^{n-1}\cos x + (n-1)\int (\sin x)^{n-2} \, dx$$

$$\int (\sin x)^n \, dx = -\frac{1}{n}(\sin x)^{n-1}\cos x + \frac{n-1}{n}\int (\sin x)^{n-2} \, dx$$

The last equation is the recursion formula. It reduces the integral of a power of sine to some stuff $(-\frac{1}{n}(\sin x)^{n-1}\cos x)$ plus the integral of a power of sine that is smaller by 2.

Here's how the formula would apply if $n = 3$:

$$\int (\sin x)^3 \, dx = -\frac{1}{3}(\sin x)^2\cos x + \frac{2}{3}\int \sin x \, dx = -\frac{1}{3}(\sin x)^2\cos x - \frac{2}{3}\cos x + c. \quad \square$$

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