Trigonometric Substitution

Trigonometric substitution ("trig substitution") reduces certain integrals to integrals of trig functions. The idea is to match the given integral against one of the following trig identities:

$$1 - (\sin\theta)^2 = (\cos\theta)^2$$
$$1 + (\tan\theta)^2 = (\sec\theta)^2$$
$$(\sec\theta)^2 - 1 = (\tan\theta)^2$$

(a) If the integral contains an expression of the form $a^2 - x^2$, try a substitution based on the first identity: $x = a\sin\theta$.

(b) If the integral contains an expression of the form $a^2 + x^2$, try a substitution based on the second identity: $x = a\tan\theta$.

(c) If the integral contains an expression of the form $x^2 - a^2$, try a substitution based on the third identity: $x = a\sec\theta$.

If you don't obtain one of the identities above after substituting, you've probably used the wrong substitution.

Example. Compute $\displaystyle\int \frac{1}{(1-x^2)^{5/2}}\,dx$.

The expression "$1 - x^2$" leads me to try

$$x = \sin\theta, \quad \text{so} \quad dx = \cos\theta\,d\theta.$$

Plug in:

$$\int \frac{1}{(1-x^2)^{5/2}}\,dx = \int \frac{1}{[1-(\sin\theta)^2]^{5/2}}\cdot\cos\theta\,d\theta = \int \frac{1}{[(\cos\theta)^2]^{5/2}}\cdot\cos\theta\,d\theta = \int \frac{1}{(\cos\theta)^5}\cdot\cos\theta\,d\theta =$$

$$\int \frac{1}{(\cos\theta)^4}\,d\theta = \int (\sec\theta)^4\,d\theta = \int (\sec\theta)^2(\sec\theta)^2\,d\theta = \int [1+(\tan\theta)^2](\sec\theta)^2\,d\theta = \int (1+u^2)\,du =$$

$$\left[u = \tan\theta, \quad du = (\sec\theta)^2\,d\theta, \quad d\theta = \frac{du}{(\sec\theta)^2}\right]$$

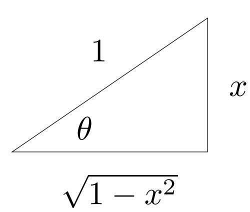
$$u + \frac{1}{3}u^3 + c = \tan\theta + \frac{1}{3}(\tan\theta)^3 + c.$$

To put the x back, I draw a right triangle. The substitution was

$$\sin\theta = x = \frac{x}{1}.$$

Since sine is the opposite side divided by the hypotenuse, I get:

$$[x = \sin\theta, \quad dx = \cos\theta\,d\theta]$$



I found the adjacent side $\sqrt{1-x^2}$ using Pythagoras. From the triangle, I see that $\tan\theta = \dfrac{x}{\sqrt{1-x^2}}$. Plugging this back in, I get

$$\tan\theta + \frac{1}{3}(\tan\theta)^3 + c = \frac{x}{\sqrt{1-x^2}} + \frac{1}{3}\frac{x^3}{(1-x^2)^{3/2}} + c. \quad \square$$

Example. Compute $\displaystyle\int (4-x^2)^{3/2}\,dx$.

To "match" the "4" in "$4-x^2$", I use $x = 2\sin\theta$ (since $2^2 = 4$). Differentiation gives $\dfrac{dx}{d\theta} = 2\cos\theta$, so $dx = 2\cos\theta\,d\theta$. I plug in and simplify:

$$\int (4-x^2)^{3/2}\,dx = \int \left(4 - 4(\sin\theta)^2\right)^{3/2} (2\cos\theta)\,d\theta =$$

$$[x = 2\sin\theta, \quad dx = 2\cos\theta\,d\theta]$$

$$\int \left(4(\cos\theta)^2\right)^{3/2} (2\cos\theta)\,d\theta = 16\int (\cos\theta)^4\,d\theta.$$

I have an even power of cosine, so I need to use the double angle formula for $(\cos\theta)^2$:

$$16\int (\cos\theta)^4\,d\theta = 16\int \left((\cos\theta)^2\right)^2\,d\theta = 16\int \left(\frac{1}{2}(1+\cos 2\theta)\right)^2\,d\theta = 4\int \left(1 + 2\cos 2\theta + (\cos 2\theta)^2\right)\,d\theta =$$

$$4\int \left(1 + 2\cos 2\theta + \left(\frac{1}{2}(1+\cos 4\theta)\right)\right)\,d\theta = 4\left(\theta + \sin 2\theta + \frac{1}{2}(\theta + \frac{1}{4}\sin 4\theta)\right) + C =$$

$$6\theta + 4\sin 2\theta + \frac{1}{2}\sin 4\theta + C.$$

I need to put the x's back.

For the terms $4\sin 2\theta$ and $\dfrac{1}{2}\sin 4\theta$, I need to express everything in terms of trig functions of θ (as opposed to 2θ or 4θ). I use the **double angle formulas** for sine:

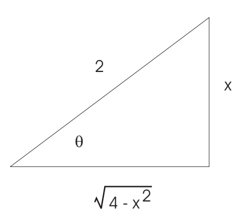
$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\sin 4\theta = 2\sin 2\theta\cos 2\theta = 2(2\sin\theta\cos\theta)\left(2(\cos\theta)^2 - 1\right) = 4\sin\theta\cos\theta\left(2(\cos\theta)^2 - 1\right)$$

Therefore,

$$\int (4-x^2)^{3/2}\,dx = 6\theta + 8\sin\theta\cos\theta + 2\sin\theta\cos\theta\left(2(\cos\theta)^2 - 1\right) + C = 6\theta + 6\sin\theta\cos\theta + 4(\cos\theta)^3\sin\theta + C.$$

The "θ" in the first term is not inside a trig function. For that term, $x = 2\sin\theta$ gives $\sin\theta = \dfrac{x}{2}$, so $\theta = \sin^{-1}\dfrac{x}{2}$.

For the second and third terms, draw a right triangle which shows the substitution.



The triangle shows $\sin\theta = \dfrac{x}{2}$ — the opposite side is x and the hypotenuse is 2 — and by Pythagoras the third side is $\sqrt{4-x^2}$. Therefore,

$$\sin\theta = \frac{x}{2} \quad \text{and} \quad \cos\theta = \frac{\sqrt{4-x^2}}{2}.$$

Plugging all of this into the last expression, I have

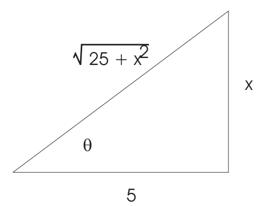
$$\int (1-x^2)^{3/2}\,dx = 6\theta + 6\sin\theta\cos\theta + 4(\cos\theta)^3\sin\theta + C = 6\sin^{-1}\frac{x}{2} + \frac{3}{2}x\sqrt{4-x^2} + \frac{1}{4}x(4-x^2)^{3/2} + C. \quad \Box$$

Example. Compute $\displaystyle\int \frac{dx}{\sqrt{25+x^2}}$.

$25+x^2$ looks like $1+(\tan\theta)^2$, so let $x = 5\tan\theta$. Then $dx = 5(\sec\theta)^2\,d\theta$, so

$$\int \frac{dx}{\sqrt{25+x^2}} = \int \frac{5(\sec\theta)^2\,d\theta}{\sqrt{25+25(\tan\theta)^2}} = \int \frac{5(\sec\theta)^2\,d\theta}{\sqrt{25(\sec\theta)^2}} = \int \frac{5(\sec\theta)^2\,d\theta}{5\sec\theta} = \int \sec\theta\,d\theta =$$

$$\left[x = 5\tan\theta, \quad dx = 5(\sec\theta)^2\,d\theta\right]$$



$$\ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{\sqrt{25+x^2}}{5} + \frac{x}{5}\right| + C. \quad \Box$$

Example. Compute $\displaystyle\int \frac{x\,dx}{\sqrt{25+x^2}}$.

This could be done using $x = 5\tan\theta$. But it's easier to do a u-substitution:

$$\int \frac{x\,dx}{\sqrt{25+x^2}} = \int \frac{x \cdot \dfrac{du}{2x}}{\sqrt{u}} = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \sqrt{u} + C = \sqrt{25+x^2} + C.$$

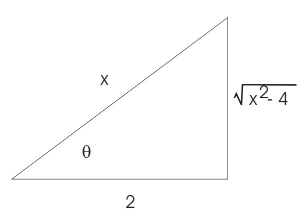
$$\left[u = 25+x^2, \quad du = 2x\,dx, \quad dx = \frac{du}{2x}\right] \quad \Box$$

Example. Compute $\displaystyle\int \sqrt{x^2-4}\,dx$.

x^2-4 looks like $(\sec\theta)^2 - 1$, so let $x = 2\sec\theta$. Then $dx = 2\sec\theta\tan\theta\,d\theta$, and

$$\int \sqrt{x^2-4}\,dx = \int \sqrt{4(\sec\theta)^2-4}(2\sec\theta\tan\theta\,d\theta) = \int \sqrt{4(\tan\theta)^2}(2\sec\theta\tan\theta\,d\theta) =$$

$$[x = 2\sec\theta, \quad dx = 2\sec\theta\tan\theta\,d\theta]$$



$$\int (2\tan\theta)(2\sec\theta\tan\theta\,d\theta) = 4\int \sec\theta(\tan\theta)^2\,d\theta = 4\int \sec\theta\left((\sec\theta)^2 - 1\right)\,d\theta =$$

$$4\int (\sec\theta)^3\,d\theta - 4\int \sec\theta\,d\theta = 2\sec\theta\tan\theta + 2\ln|\sec\theta + \tan\theta| - 4\ln|\sec\theta + \tan\theta| + C =$$

$$2\sec\theta\tan\theta - 2\ln|\sec\theta + \tan\theta| = \frac{1}{2}x\sqrt{x^2 - 4} - 2\ln\left|\frac{x}{2} + \frac{\sqrt{x^2 - 4}}{2}\right| + C. \quad \square$$
