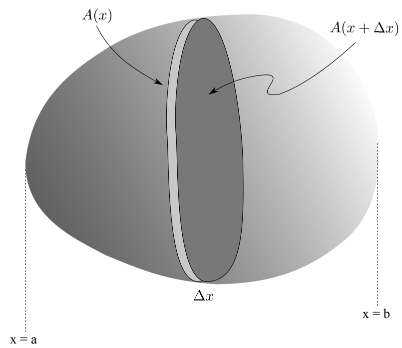
Volumes by Cross Sections

Suppose a solid extends from $x = a$ to $x = b$. Suppose that when it is cut by planes perpendicular to the x-axis, the cross-section of the solid cut by such a plane has area $A(x)$. As usual, I divide the interval from a to b into subintervals of width Δx (say $\Delta x = \dfrac{b-a}{n}$ for some n).



On a typical subinterval, I have cross-sections of areas $A(x)$ and $A(x+\Delta x)$. It's reasonable to suppose that if the function $A(x)$ is "nice enough", there should be a number $\overline{x}$ between x and $x+\Delta x$ such that the volume of the small cross-section (or "slice") of thickness Δx from x to $x+\Delta x$ is exactly

$$A(\overline{x}) \cdot \Delta x.$$

Adding up the volumes of such cross-sections gives the volume of the solid:

$$V = \sum A(\overline{x}) \cdot \Delta x.$$

Replacing $A(\overline{x})$ with $A(x)$ only gives an approximation:

$$V \approx \sum A(x) \cdot \Delta x.$$

But if I take the limit as $\Delta x \to 0$, then if $A(x)$ is "nice enough" (for example, continuous as a function of x), then in the limit I will get the exact volume. It will be given by

$$V = \lim_{\Delta x \to 0} \sum A(x) \cdot \Delta x = \int_a^b A(x)\,dx.$$

Example. The cross-sections of a solid in planes perpendicular to the x-axis have area

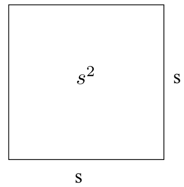
$$A(x) = 6x^2 + 5.$$

Find the volume of the solid from $x = 0$ to $x = 1$.

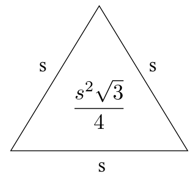
Since the cross-sectional area function is given, I just integrate from 0 to 1:

$$V = \int_0^1 (6x^2+5)\,dx = \left[2x^3 + 5x\right]_0^1 = 7. \quad \square$$

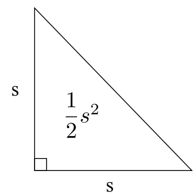
In the problems that follow, you need to determine the cross-sectional area function. In many cases, it comes from an area formula from geometry. Here are some common ones.



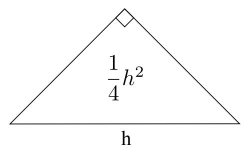
a square with sides of length s



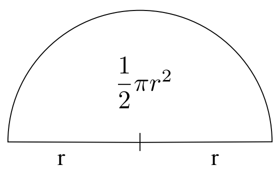
an equilateral triangle with sides of length s



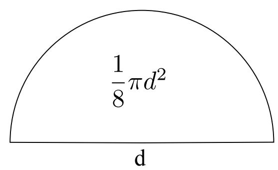
an isosceles right triangle with legs of length s



an isosceles right triangle with hypotenuse of length h

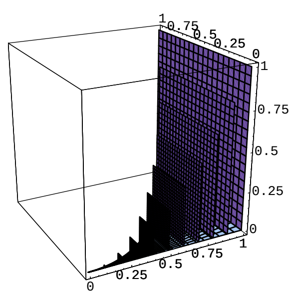


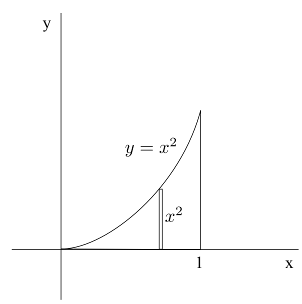
a semicircle of radius r



a semicircle of diameter d

Example. The base of a solid is the region in the x-y plane bounded above by $y = x^2$ and below by the x-axis, from $x = 0$ to $x = 1$. The cross-sections in planes perpendicular to the x-axis are squares with one side lying in the x-y plane. Find the volume of the solid.



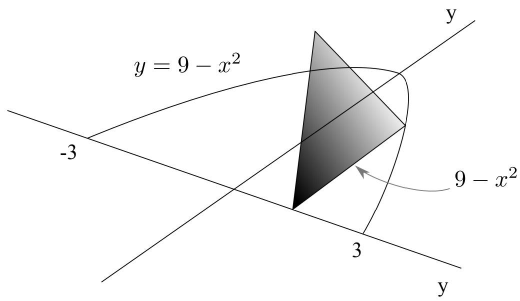


The first picture shows some of the square cross-sections. The second picture shows the base of the solid, with the edge of one of the square cross-sections drawn. The edge of the square is x^2, so the area of the cross-section is $(x^2)^2$. The volume is

$$V = \int_0^1 (x^2)^2\,dx = \int_0^1 x^4\,dx = \left[\frac{1}{5}x^5\right]_0^1 = \frac{1}{5}. \quad \square$$

Example. The base of a solid is the region in the x-y plane bounded above by $y = 9 - x^2$ and below by the x-axis. The cross-sections in planes perpendicular to the x-axis are equilateral triangles with one side lying in the x-y plane. Find the volume of the solid.

The base is bounded by $y = 9 - x^2$ and the x-axis. The parabola intersects the x-axis at $x = -3$ and $x = 3$.



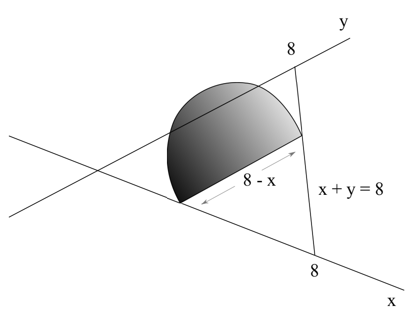
The picture shows a typical cross-section. It's an equilateral triangle, and its side has length $9 - x^2$. Hence, the area of the cross-section is $\dfrac{\sqrt{3}}{4}(9 - x^2)^2$.

The volume is

$$V = \int_{-3}^{3} \frac{\sqrt{3}}{4}(9 - x^2)^2\,dx = \frac{\sqrt{3}}{4}\int_{-3}^{3}(81 - 18x^2 + x^4)\,dx = \frac{\sqrt{3}}{4}\left[81x - 6x^3 + \frac{1}{5}x^5\right]_{-3}^{3} =$$

$$\frac{324\sqrt{3}}{5} = 112.23689\ldots. \quad \square$$

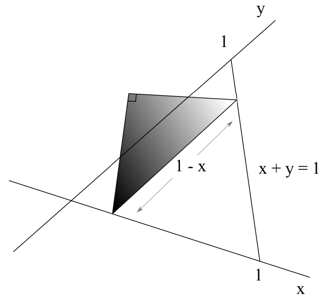
Example. The base of a solid is the region in the first quadrant cut off by the line $x + y = 8$. The cross-sections in planes perpendicular to the x-axis are semicircles with their diameters lying in the x-y plane. Find the volume of the solid.



The diameter of a typical cross-section is $y = 8 - x$, so the radius is $\dfrac{8 - x}{2}$. The volume is

$$V = \int_{0}^{8} \frac{\pi}{2}\left(\frac{8 - x}{2}\right)^2 dx = \frac{\pi}{8}\int_{0}^{8}(8 - x)^2\,dx = \frac{\pi}{8}\left[-\frac{1}{3}(8 - x)^3\right]_{0}^{8} = \frac{64\pi}{3} = 67.02064\ldots. \quad \square$$

Example. The base of a solid is the region in the first quadrant cut off by the line $x + y = 1$. The cross-sections in planes perpendicular to the x-axis are isosceles right triangles with the hypotenuses lying in the x-y plane. Find the volume of the solid.



Since the hypotenuse of a typical triangle is $1 - x$, the side of such a triangle is $\dfrac{1-x}{\sqrt{2}}$. The area of a triangular slice is one-half the base times the height, which is

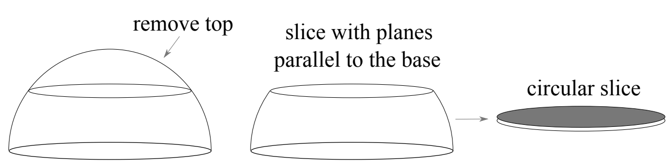
$$\frac{1}{2}\left(\frac{1-x}{\sqrt{2}}\right)\left(\frac{1-x}{\sqrt{2}}\right) = \frac{1}{4}(1-x)^2.$$

The volume is

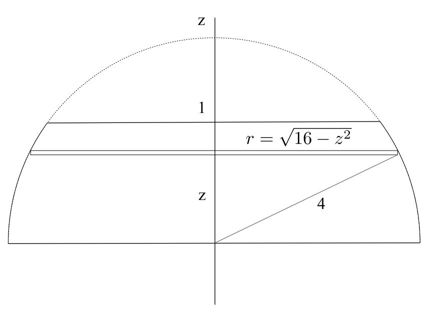
$$V = \frac{1}{4}\int_0^1 (1-x)^2\,dx = \frac{1}{4}\left[-\frac{1}{3}(1-x)^3\right]_0^1 = \frac{1}{12}. \quad \square$$

Example. A solid hemisphere of radius 4 has its base in the x-y-plane. It is cut by a plane parallel to the x-y-plane and 1 unit above it. Find the volume of the part of the hemisphere which lies between the cutting plane and the x-y-plane.

In this problem, you have to decide how to slice the solid in order to give cross-sections whose areas you can compute. Slicing the solid by planes parallel to the x-y plane produces circular disks.



The next picture shows the solid in cross-section, with a typical slice drawn.



By Pythagoras' theorem, the radius of a disk lying z units above the x-y-plane is $\sqrt{16 - z^2}$, so its cross-sectional area is $\pi(16 - z^2)$.

The volume is

$$\int_0^1 \pi(16 - z^2)\,dz = \pi\left[16z - \frac{1}{3}z^3\right]_0^1 = \frac{47\pi}{3} = 49.21828\ldots. \quad \square$$
