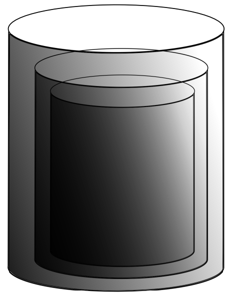
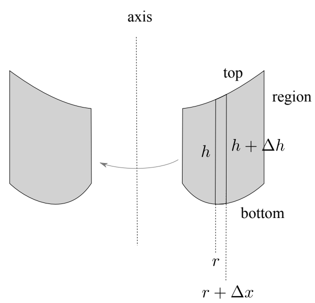
Volumes by Cylindrical Shells

In this section, we'll consider another method for computing the volume of a solid of revolution. The idea is to break up the solid into **cylindrical shells**. The solid will be built up of an "infinite number" of cylinders, nested inside one another.



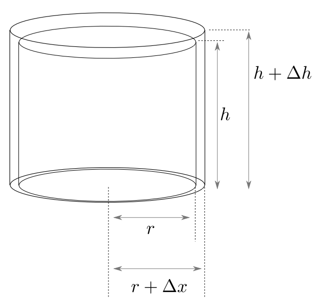
Suppose the region being revolved is bounded by a top curve and a bottom curve, over some interval on the x-axis. Consider a small piece of the region between the top and bottom curves and extending horizontally from r to $r + \Delta x$:



r is the distance from the axis of revolution to the edge of the small piece. It might not be the same as x if the axis of revolution is not the x-axis.

The height of the small piece varies from h to $h + \Delta h$. Note that Δh could be positive, negative, or 0. For any x-coordinate, this height is just the top curve minus the bottom curve.

As the small piece revolves about the axis of revolution, it sweeps out a solid which is approximately the region between two cylinders:



We can approximate the volume of this region as follows. The cross-sectional area perpendicular to the axis of revolution is the area between two circles of radii r and $r + \Delta x$:

$$\pi[(r + \Delta x)^2 - r^2] = 2\pi r \Delta x + \pi(\Delta x)^2.$$

For the height, I'll use $\overline{h}$, the average value of h over the interval.

The volume of the solid swept out by the small piece is

$$(2\pi r \Delta x + \pi(\Delta x)^2) \cdot \overline{h}.$$

If I sum the volumes of these solids and let the length of the interval Δx go to 0, I should get the volume of the solid of revolution.

As $\Delta x \to 0$, I can neglect the term $\pi(\Delta x)^2$, since it is small compared to the other term $2\pi r \Delta x$. In addition, $\overline{h} \to h$, where h is just the difference between the top and bottom curves. So I have a sum of the form

$$\sum 2\pi r h \Delta x.$$

In the limit as $\Delta x \to 0$, this gives a Riemann sum for the integral

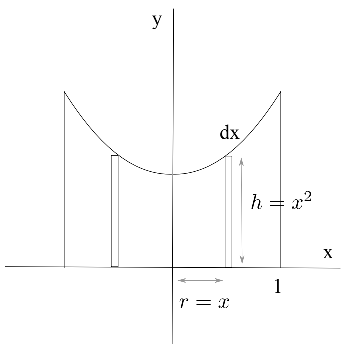
$$\int_a^b 2\pi r h\, dx.$$

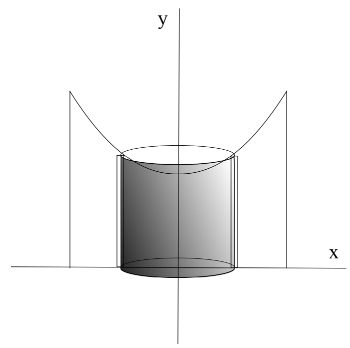
If the axis of revolution is horizontal (parallel to the x-axis), the formula is

$$\int_a^b 2\pi r h\, dy.$$

Example. The area under $y = x^2 + 1$ from $x = 0$ to $x = 1$ is revolved about the y-axis. Find the volume generated.

To do these problems I'll draw a cross-sectional picture. It shows the original region, and a copy of the region revolved to the other side of the axis. I'll draw a typical shell, showing the two places where the shell "goes through" the x-y-plane as vertical rectangles.





The first picture shows the cross-sectional picture. The second picture shows the cylinder drawn in the cross-section.

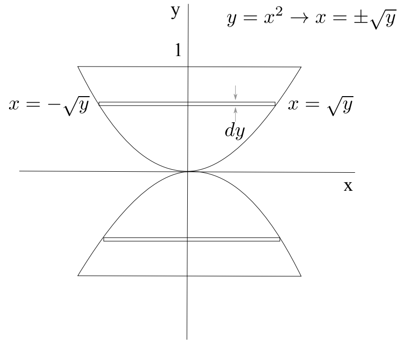
In the future, I'll just draw the cross-sectional picture.

The height of a typical cylindrical shell is $h = x^2 + 1$. The radius of a typical cylindrical shell is $r = x$. The volume of the solid is

$$V = \int_0^1 2\pi x(x^2+1)\, dx = 2\pi \int_0^1 (x^3 + x)\, dx = 2\pi \left[\frac{1}{4}x^4 + \frac{1}{2}x^2\right]_0^1 = \frac{3\pi}{2} \approx 4.71239. \quad \square$$

Example. The region bounded by $y = x^2$ and $y = 1$ is revolved about the x-axis. Find the volume of the solid generated.

In this case, the shells go "sideways", parallel to the x-axis:



The height of a typical cylindrical shell is

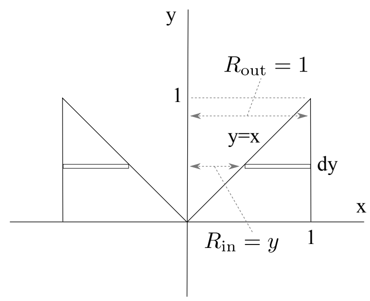
$$h = (\text{right}) - (\text{left}) = \sqrt{y} - (-\sqrt{y}) = 2\sqrt{y}.$$

The radius of a typical cylindrical shell is $r = y$, the distance from the axis (the x-axis) to the side of the shell. The volume of the solid is

$$V = \int_0^1 2\pi y \cdot 2\sqrt{y}\, dy = 4\pi \int_0^1 y^{3/2}\, dy = 4\pi \left[\frac{2}{5} y^{5/2}\right]_0^1 = \frac{8\pi}{5} = 5.02654\ldots. \quad \square$$

Example. The area under $y = x$ from $x = 0$ to $x = 1$ is revolved about the y-axis. Find the volume of the solid generated using: (a) Circular washers, and (b) Cylindrical shells.

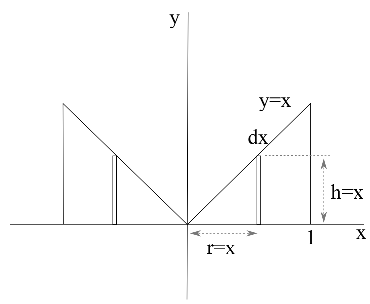
(a) Using washers, the inner radius of a typical washer is y and the outer radius of a typical washer is 1.



The volume of the solid is

$$V = \int_0^1 \pi(1^2 - y^2)\, dy = \pi \left[y - \frac{1}{3} y^3\right]_0^1 = \frac{2\pi}{3} = 2.09439\ldots.$$

(b) Using cylindrical shells, the height of a typical cylindrical shell is $h = x$ and the radius of a typical cylindrical shell is $r = x$.

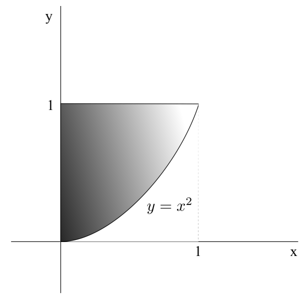


The volume of the solid is

$$V = \int 2\pi x \cdot x\, dx = 2\pi \left[\frac{1}{3} x^3\right]_0^1 = \frac{2\pi}{3} = 2.09439\ldots.$$

Note that the thickness of a typical shell is dx, while the thickness of a typical washer is dy. $\square$

Example. Let R be the region bounded by $y = x^2$ and $y = 1$, from $x = 0$ to $x = 1$.



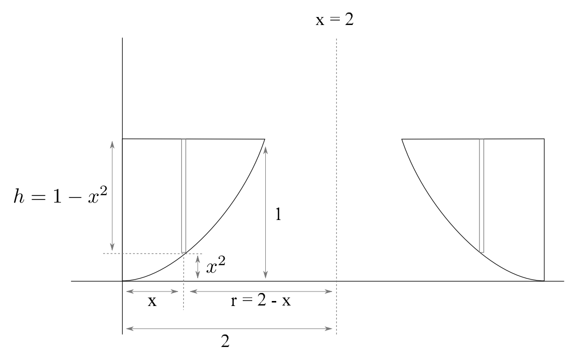
(a) Find the volume generated by revolving R about the line $x = 2$.

(b) Find the volume generated by revolving R about the line $x = -5$.

(c) Find the volume generated by revolving R about the line $y = -2$.

(d) Find the volume generated by revolving R about the line $y = 2$.

(a) The height of a typical cylindrical shell is $h = 1 - x^2$. The radius of a typical cylindrical shell is $r = 2 - x$.



The radius is the distance from the axis to the side of a typical shell. In this case, I found it by subtraction.

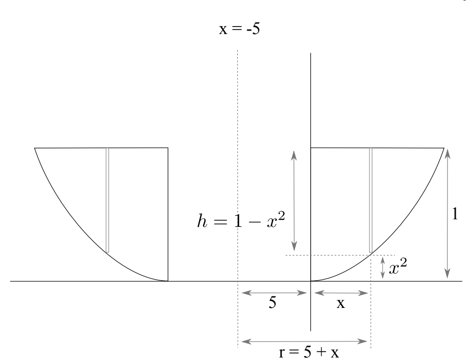
Notice that I'm not trying to draw the cross-sectional picture to scale. Also, when I figure out r and h, I do my work on the original copy of the region, not the rotated copy. The rotated copy just gives me a sense of what the volume of revolution looks like.

The volume of the solid is

$$V = \int_0^1 2\pi(2 - x)(1 - x^2)\,dx = 2\pi \int_0^1 (2 - x - 2x^2 + x^3)\,dx = 2\pi \left[2x - \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4\right]_0^1 =$$

$$\frac{13\pi}{6} = 6.80678\ldots. \quad \square$$

(b) The height of a typical cylindrical shell is $h = 1 - x^2$. The radius of a typical cylindrical shell is $r = 5 + x$.



The radius is the distance from the axis to the side of a typical shell. In this case, I found it by addition. (Compare this to what I did in part (a).)

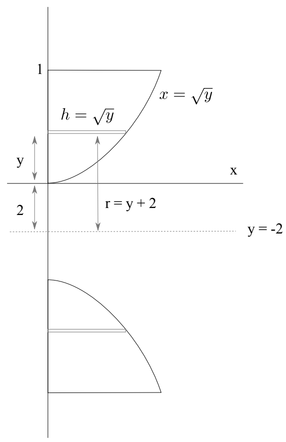
Once again, the picture is not drawn to scale.

The volume of the solid is

$$V = \int_0^1 2\pi(5+x)(1-x^2)\,dx = 2\pi\int_0^1 (5+x-5x^2-x^3)\,dx = 2\pi\left[5x+\frac{1}{2}x^2-\frac{5}{3}x^3-\frac{1}{4}x^4\right]_0^1 = \frac{43\pi}{6} = 22.51474\ldots. \quad \square$$

(c) If I solve $y = x^2$ for x, I get $x = \sqrt{y}$. I use the positive square root, because I'm considering the part of the curve in the first quadrant where x is positive.

The height of a typical cylindrical shell is $h = \sqrt{y}$. The radius of a typical cylindrical shell is $r = y + 2$.

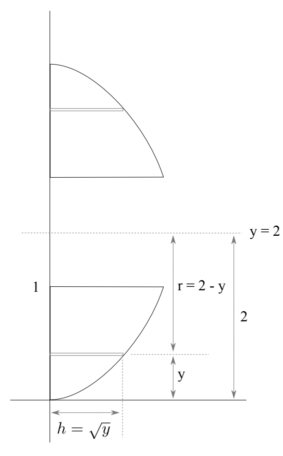


The volume of the solid is

$$V = \int_0^1 2\pi(y+2)\sqrt{y}\,dy = 2\pi\int_0^1 (y^{3/2}+2y^{1/2})\,dy = 2\pi\left[\frac{2}{5}y^{5/2}+\frac{4}{3}y^{3/2}\right]_0^1 = \frac{52\pi}{15} = 10.89085\ldots. \quad \square$$

You might try to set this problem up using circular washers.

(d) The height of a typical cylindrical shell is $h = \sqrt{y}$. The radius of a typical cylindrical shell is $r = 2 - y$.



The volume of the solid is

$$V = \int_0^1 2\pi(2-y)\sqrt{y}\,dy = 2\pi\int_0^1 (2y^{1/2}-y^{3/2})\,dy = 2\pi\left[\frac{4}{3}y^{3/2}-\frac{2}{5}y^{5/2}\right]_0^1 = \frac{28\pi}{15} = 5.86430\ldots. \quad \square$$