Arc Length in R^n

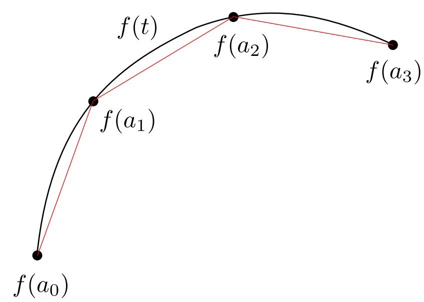
Let $f : [a, b] \to \mathbb{R}^n$ be a curve in $\mathbb{R}^n$. How would you find the length of the curve? One approach is to start by approximating the curve with segments.

Divide the base interval $[a, b]$ up into subintervals:

$$a_0 = a,\ a_1,\ a_2,\ \ldots a_{n-1},\ a_n = b.$$

This is called a **partition** of the subinterval. You may recall doing this to set up **Riemann sums**.

If you plug the a's into f, you get points on the curve which you can connect with segments. Here's a pictures with 4 points and 3 segments:



The sum of the segments lengths approximates the length of the curve.

Definition. A curve $f : [a, b] \to \mathbb{R}^n$ is **rectifiable** if the sums of the segment lengths have an **upper bound**: that is, there is a number M such that for every partitition of $[a, b]$, the sum of the segment lengths is less than M.

If a curve is rectifiable, the **length** of the curve is least upper bound of the numbers M which bound the sums of segment lengths for all partititions.

If a curve has reasonable properties, we can compute the length using an integral.

Theorem. Suppose $f : [a, b] \to \mathbb{R}^n$ is a curve where f is differentiable and $f'(t)$ is continuous. Then f is rectifiable, and the length of f is

$$\int_a^b \|f'(t)\|\, dt. \quad \square$$

While the proof is a little technical, we can see why this makes sense. $\|f'(t)\|$ is the **speed** of an object moving along the curve. In a small increment Δt of time, the object moves a distance $\|f'(t)\|\,\Delta t$. If we let the time increments go to 0 and add up the distances by integrating, we get the distance travelled by the object, which is the length of the curve.

Example. Find the length of the curve

$$x = t^2 - t, \quad y = \sqrt{3}t^2, \quad z = \frac{2\sqrt{12}}{3}t^{3/2} + 1, \quad \text{for} \quad 0 \le t \le 1.$$

$$\frac{dx}{dt} = 2t - 1, \quad \frac{dy}{dt} = 2\sqrt{3}t, \quad \frac{dz}{dt} = \sqrt{12}t^{1/2}.$$

$$\left(\frac{dx}{dt}\right)^2 = 4t^2 - 4t + 1, \quad \left(\frac{dy}{dt}\right)^2 = 12t^2, \quad \left(\frac{dz}{dt}\right)^2 = 12t.$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = 16t^2 + 8t + 1 = (4t + 1)^2.$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = 4t + 1.$$

The length is

$$\int_0^1 (4t+1)\,dt = \left[2t^2 + t\right]_0^1 = 3. \quad \Box$$

Example. Find the length of the curve

$$x = e^{2t}, \quad y = e^{-2t}, \quad z = \sqrt{8}t, \quad \text{for} \quad 0 \le t \le 1.$$

$$\frac{dx}{dt} = 2e^{2t}, \quad \frac{dy}{dt} = -2e^{-2t}, \quad derzt = \sqrt{8}.$$

$$\left(\frac{dx}{dt}\right)^2 = 4e^{4t}, \quad \left(\frac{dy}{dt}\right)^2 = 4e^{-4t}, \quad \left(\frac{dz}{dt}\right)^2 = 8.$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = 4e^{4t} + 8 + 4e^{-4t} = 4(e^{4t} + 2 + e^{-4t}) = 4(e^{2t} + e^{-2t})^2.$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = 2(e^{2t} + e^{-2t}).$$

The length is

$$\int_0^1 2(e^{2t} + e^{-2t})\,dt = \left[e^{2t} - e^{-2t}\right]_0^1 = e^2 - e^{-2} = 7.25372\ldots. \quad \Box$$
