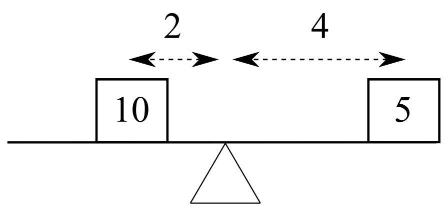
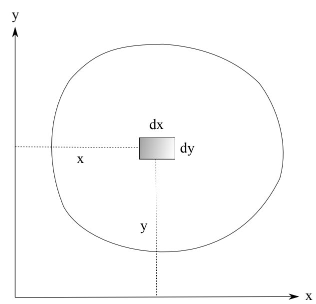
Center of Mass

Imagine two weights on opposite sides of a balance board. One weight is 5 kilograms and is 4 meters from the center. The other weight is 10 kilograms and is 2 meters from the center.



The weights balance. What is relevant is the products of the mass and the distance of the mass from the center.

Consider region R in the plane. Imagine that it is made out of a thin material with varying density $\delta(x, y)$. Consider a small rectangular piece with dimensions dx by dy located at the point (x, y). Its mass is $\delta(x, y)\, dx\, dy$.



The total mass is

$$M = \iint_R \delta(x, y)\, dx\, dy.$$

By analogy with the balance board example, I measure the "twisting" about the y-axis produced by the small rectangular piece. It is the product of the mass $\delta(x, y)\, dx\, dy$ and the distance to the y-axis, which is x:

$$x \cdot d(x, y)\, dx\, dy.$$

The total amount of"twisting" about the y-axis is the x-moment, and is obtained by integrating ("adding up") the "twisting" produced by each of the small pieces that make up the region:

$$m_x = \iint_R x\delta(x, y)\, dx\, dy.$$

I can define the y-moment in the same way:

$$m_y = \iint_R y\delta(x, y)\, dx\, dy.$$

Now imagine the region compressed to small ball which has the same total mass M as the original region. Where should the small ball be located so that it produces the same x and y-moments? The location is called the **center of mass** of the original region; if its coordinates are $(\overline{x}, \overline{y})$, I want

$$\overline{x} \cdot M = m_x \quad \text{and} \quad \overline{y} \cdot M = m_y.$$

Solving for $\overline{x}$ and $\overline{y}$, I get

$$\overline{x} = \frac{m_x}{M} = \frac{\displaystyle\iint_R x\delta(x,y)\,dx\,dy}{\displaystyle\iint_R \delta(x,y)\,dx\,dy},$$

$$\overline{y} = \frac{m_y}{M} = \frac{\displaystyle\iint_R y\delta(x,y)\,dx\,dy}{\displaystyle\iint_R \delta(x,y)\,dx\,dy}.$$

We can do the same thing in 3 dimensions. Suppose a solid object occupies a region R in space, and that the density of the solid at the point (x,y,z) is $\delta(x,y,z)$. The total mass of the object is given by

$$M = \iiint_R \delta(x,y,z)\,dx\,dy\,dz.$$

The **moments** in the x, y, and z directions are given by

$$m_x = \iiint_R x\delta(x,y,z)\,dx\,dy\,dz,$$

$$m_y = \iiint_R y\delta(x,y,z)\,dx\,dy\,dz,$$

$$m_z = \iiint_R z\delta(x,y,z)\,dx\,dy\,dz.$$

You can think of them in a rough way as representing the "twisting" about the axis in question produced by a small bit of mass $\delta(x,y,z)\,dx\,dy\,dz$ at the point (x,y,z).

The **center of mass** is the point $(\overline{x},\overline{y},\overline{z})$ given by

$$\overline{x} = \frac{m_x}{M} = \frac{\displaystyle\iiint_R x\delta(x,y,z)\,dx\,dy\,dz}{\displaystyle\iiint_R \delta(x,y,z)\,dx\,dy\,dz},$$

$$\overline{y} = \frac{m_y}{M} = \frac{\displaystyle\iiint_R y\delta(x,y,z)\,dx\,dy\,dz}{\displaystyle\iiint_R \delta(x,y,z)\,dx\,dy\,dz},$$

$$\overline{z} = \frac{m_z}{M} = \frac{\displaystyle\iiint_R z\delta(x,y,z)\,dx\,dy\,dz}{\displaystyle\iiint_R \delta(x,y,z)\,dx\,dy\,dz}.$$

If a region in the plane or a solid in space has constant density $\delta(x,y,z) = k$, then the center of mass is called the **centroid**. In this case, the density drops out of the formulas for $\overline{x}$, $\overline{y}$, and $\overline{z}$. For example,

$$\overline{x} = \frac{\displaystyle\iiint_R x\delta(x,y,z)\,dx\,dy\,dz}{\displaystyle\iiint_R \delta(x,y,z)\,dx\,dy\,dz} = \frac{\displaystyle\iiint_R x\cdot k\,dx\,dy\,dz}{\displaystyle\iiint_R k\,dx\,dy\,dz} = \frac{\displaystyle\iiint_R x\,dx\,dy\,dz}{\displaystyle\iiint_R dx\,dy\,dz}.$$

(Of course, you can usually use a double integral to compute the volume of a solid.)

The **centroid** of the region is $(\overline{x}, \overline{y}, \overline{z})$, where

$$\overline{x} = \frac{\iiint_R x\,dx\,dy\,dz}{\iiint_R dx\,dy\,dz}, \quad \overline{y} = \frac{\iiint_R y\,dx\,dy\,dz}{\iiint_R dx\,dy\,dz}, \quad \overline{z} = \frac{\iiint_R z\,dx\,dy\,dz}{\iiint_R dx\,dy\,dz}.$$

Recall that if R is a region in space, the volume of R is

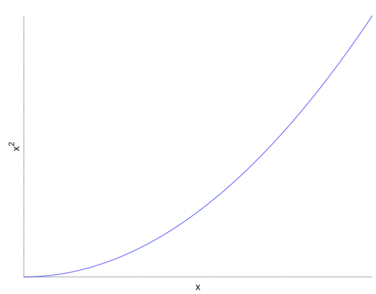
$$V = \iiint_R dx\,dy\,dz.$$

Thus, the denominators of the fractions above are all equal to **volume** of R.
The corresponding formulas for the centroid of a region in the plane are:

$$\overline{x} = \frac{\iint_R x\,dx\,dy}{\iint_R dx\,dy}, \quad \overline{y} = \frac{\iint_R y\,dx\,dy}{\iint_R dx\,dy}$$

Notice that the integral $\iint_R dx\,dy$ is just the **area** of R.

Example. Find the centroid of the region in the first quadrant bounded above by $y = x^2$, from $x = 0$ to $x = 1$.



Since the question is asking for the centroid, the density is assumed to be constant.
The region is

$$\left\{ \begin{matrix} 0 \le x \le 1 \\ 0 \le y \le x^2 \end{matrix} \right\}$$

First, the area is

$$\int_0^1 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}.$$

Note that I didn't need a double integral to find the area.
The x-moment is

$$\int_0^1 \int_0^{x^2} x\,dx\,dx = \int_0^1 x\,[y]_0^{x^2}\,dx = \int_0^1 x^3\,dx = \left[\frac{1}{4}x^4\right]_0^1 = \frac{1}{4}.$$

The y-moment is

$$\int_0^1 \int_0^{x^2} y\,dx\,dx = \int_0^1 \left[\frac{1}{2}y^2\right]_0^{x^2}\,dx = \frac{1}{2}\int_0^1 x^4\,dx = \left[\frac{1}{10}x^5\right]_0^1 = \frac{1}{10}.$$

Therefore,

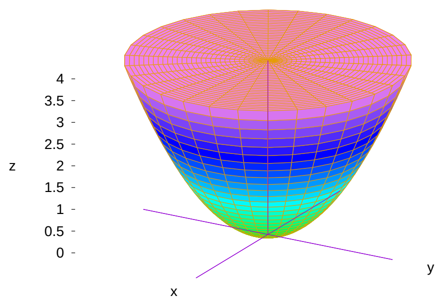
$$\overline{x} = \frac{\dfrac{1}{4}}{\dfrac{1}{3}} = \frac{3}{4}.$$

$$\overline{y} = \frac{\dfrac{1}{10}}{\dfrac{1}{3}} = \frac{3}{10}.$$

The centroid is $\left(\dfrac{3}{4}, \dfrac{3}{10}\right)$. $\square$

You can often use **symmetry** to find the coordinates of the center of mass, or to determine a relationship among the coordinates — for example, in some cases smmetry implies that some of the coordinates will be equal.

Example. Find the centroid of the region R bounded above by the plane $z = 4$ and below by the paraboloid $z = x^2 + y^2$.



By symmetry, $\overline{x} = \overline{y} = 0$, so I only need to find $\overline{z}$. I'll use cylindrical coordinates.

$z = 4$ and $z = x^2 + y^2$ intersect in $x^2 + y^2 = 4$, so the projection of R into the x-y-plane is the interior of the circle of radius 2 centered at the region.

Note that $z = x^2 + y^2 = r^2$ in cylindrical.

The region is

$$\left\{\begin{matrix} 0 \le \theta \le 2\pi \\ 0 \le r \le 2 \\ r^2 \le z \le 4 \end{matrix}\right\}$$

The volume is

$$\int_0^{2\pi} \int_0^2 (4 - r^2) \cdot r \, dr \, d\theta = 2\pi \int_0^2 (4r - r^3) \, dr = 2\pi \left[2r^2 - \frac{1}{4}r^4\right]_0^2 = 8\pi.$$

The z-moment is

$$\int_0^{2\pi} \int_0^2 \int_{r^2}^4 z \cdot r \, dz \, dr \, d\theta = 2\pi \int_0^2 r \left[\frac{1}{2}z^2\right]_{r^2}^4 dr = 2\pi \int_0^2 r \left(8 - \frac{1}{2}r^4\right) dr =$$

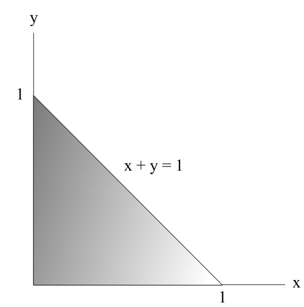
$$2\pi \int_0^2 \left(8r - \frac{1}{2}r^5\right) dr = 2\pi \left[4r^2 - \frac{1}{12}r^6\right]_0^2 = \frac{64\pi}{3}.$$

Hence,

$$\overline{z} = \frac{\dfrac{64\pi}{3}}{8\pi} = \frac{8}{3}.$$

The centroid is $\left(0, 0, \dfrac{8}{3}\right)$. □

Example. Let R be the region in the first quadrant cut off by the line $x + y = 1$. Suppose the region is made of a material with density $\delta(x, y) = 6x + 6y$. Find the coordinates of the center of mass.



The region and the density are symmetric in x and y, so $\overline{x} = \overline{y}$. I only need to find one of the coordinates. The region is

$$\left\{\begin{matrix} 0 \le x \le 1 \\ 0 \le y \le 1 - x \end{matrix}\right\}$$

The mass is

$$\int_0^1 \int_0^{1-x} (6x + 6y)\, dy\, dx = \int_0^1 \left[6xy + 3y^2\right]_0^{1-x} dx = \int_0^1 \left(6x(1-x) + 3(1-x)^2\right) dx =$$

$$\int_0^1 \left(6x - 6x^2 + 3(1-x)^2\right) dx = \left[3x^2 - 2x^3 - (1-x)^3\right]_0^1 = 2.$$

The x-moment is

$$\int_0^1 \int_0^{1-x} x(6x + 6y)\, dy\, dx = \int_0^1 \int_0^{1-x} (6x^2 + 6xy)\, dy\, dx = \int_0^1 \left[6x^2 y + 3xy^2\right]_0^{1-x} dx =$$

$$\int_0^1 \left(6x^2(1-x) + 3x(x-1)^2\right) dx = \int_0^1 (3x - 3x^3)\, dx =$$

$$\left[\frac{3}{2}x^2 - \frac{3}{4}x^4\right]_0^1 = \frac{3}{4}.$$

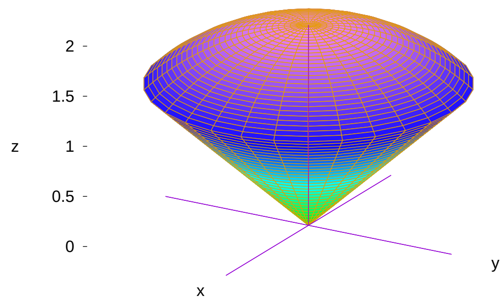
Hence,

$$\overline{x} = \frac{\frac{3}{4}}{2} = \frac{3}{8}.$$

The center of mass is $\left(\dfrac{3}{8}, \dfrac{3}{8}\right)$. □

Example. Let R be the solid bounded below by $z = \sqrt{x^2 + y^2}$ and above by $z = \sqrt{4 - x^2 - y^2}$, and assume that the density is $\delta(x, y, z) = z$. Find the coordinates of the center of mass.

I'll convert to spherical coordinates. $z = \sqrt{x^2 + y^2}$ is a cone whose sides make an angle o $\dfrac{\pi}{4}$ with the positive z-axis. $z = \sqrt{4 - x^2 - y^2}$ is the top hemisphere of a sphere of radius 2 centered at the origin.



The region is

$$\left\{\begin{matrix} 0 \le \theta \le 2\pi \\ 0 \le \rho \le 2 \\ 0 \le \phi \le \dfrac{\pi}{4} \end{matrix}\right\}$$

Note that the density is

$$\delta = z = \rho \cos \phi.$$

Hence, the mass is

$$\int_0^{2\pi} \int_0^2 \int_0^{\pi/4} \rho \cos \phi \cdot \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta = 2\pi \int_0^2 \rho^3 \left[\frac{1}{2} (\sin \phi)^2\right]_0^{\pi/4} d\rho =$$

$$\pi \int_0^2 \rho^3 \, d\rho = \pi \left[\frac{1}{4} \rho^4\right]_0^2 = 2\pi.$$

(I did the ϕ integral using the substitution $u = \sin \phi$.)

Since the region and the density are both symmetric about the z-axis, $\overline{x} = \overline{y} = 0$. Therefore, I only need to find $\overline{z}$.

Since $\delta = z = \rho \cos \phi$, the z-moment is

$$\int_0^{2\pi} \int_0^2 \int_0^{\pi/4} \rho \cos \phi \cdot \rho \cos \phi \cdot \rho^2 \sin \phi \, d\phi \, d\rho \, d\theta = 2\pi \int_0^2 \int_0^{\pi/4} \rho^4 (\cos \phi)^2 \sin \phi \, d\phi \, d\rho =$$

$$2\pi \int_0^2 \rho^4 \left[-\frac{1}{3} (\cos \phi)^3\right]_0^{\pi/4} d\rho = \frac{2\pi}{3} \left(1 - \frac{1}{2\sqrt{2}}\right) \int_0^2 \rho^4 \, d\rho =$$

$$\frac{2\pi}{3} \left(1 - \frac{1}{2\sqrt{2}}\right) \left[\frac{1}{5} \rho^5\right]_0^2 = \frac{64\pi}{15} \left(1 - \frac{1}{2\sqrt{2}}\right).$$

(I did the ϕ integral using the substitution $u = \cos \phi$.)

Hence,

$$\overline{z} = \frac{\dfrac{64\pi}{15} \left(1 - \dfrac{1}{2\sqrt{2}}\right)}{2\pi} = \frac{32}{15} \left(1 - \frac{1}{2\sqrt{2}}\right).$$

The center of mass is $\left(0, 0, \dfrac{32}{15} \left(1 - \dfrac{1}{2\sqrt{2}}\right)\right)$. □
