4-5-2018

Change of Variables

The **change of variables** formula for multiple integrals is like u-substitution for single-variable integrals. I'll give the general change of variables formula first, and consider specific cases in lower dimensions below.

Theorem. Let U be an open set in $\mathbb{R}^n$. Let $T : U \to \mathbb{R}^n$ be a function which satisfies:

(a) T is injective: If $T(x) = T(y)$, then $x = y$.

(b) T has continuous partial derivatives.

(c) The Jacobian $JT = \det DT(x)$ is nonzero for all $x \in U$.

Suppose $f : T(U) \to \mathbb{R}$ is an integrable function. Then

$$\int_{T(U)} f = \int_U (f \circ T)\ |JT|. \quad \square$$

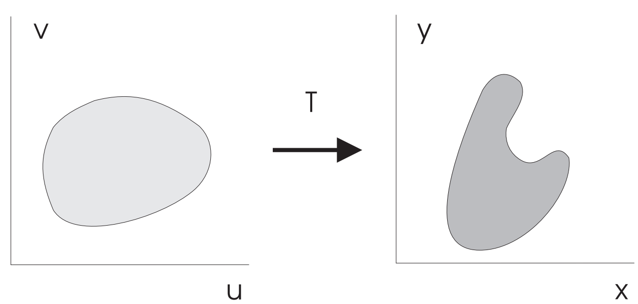
The conditions on T ensure that the change of variables is "nicely behaved" — for example, we don't have the region U doubling up on itself.

The term $f \circ T$ is the composite of f and T: You change variables in the function you're integrating, substituting using the transformation T.

The term $|\det DT|$ is roughly the factor by which volumes are "expanded" by T. For instance, in converting from rectangular to polar in a double integral, it is the factor of "r" in "$r\,dr\,d\theta$".

In this course, the important cases are in lower dimensions.

Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a coordinate transformation of the plane. Suppose T takes a region U in the u-v plane to a region R in the x-y plane.



In component form, $(x, y) = T(u, v)$ looks like

$$x = g(u, v), \quad y = h(u, v).$$

As in the theorem, I need to assume make some technical assumptions to ensure that the transformation is reasonably "nice". Specifically:

1. T is injective.

1. T has continuous first partial derivatives.

3. The Jacobian JT is nonzero on Q.

You can usually get away with some exceptions to these conditions if the set of exceptional points is "small". For example, if $JT = 0$ on a curve (technically, on a set of **content** 0) the result is still true. To avoid technicalities, I'll only consider situations in which it is okay to proceed.

Under the assumptions above, the **change of variables** formula says that

$$\iint_R f(x,y)\,dx\,dy = \iint_U f\left[x(u,v), y(u,v)\right] \cdot |JT|\,du\,dv.$$

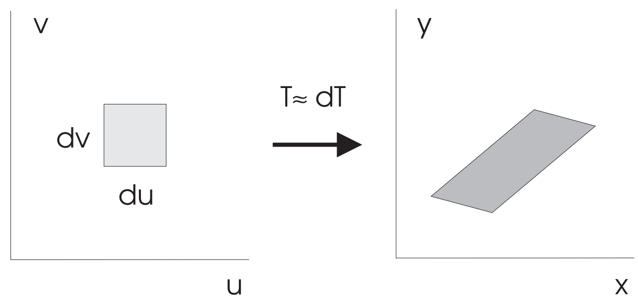
To apply the formula to compute $\displaystyle\iint_R f(x,y)\,dx\,dy$, you need to:

1. Replace the limits for R with the limits for U.

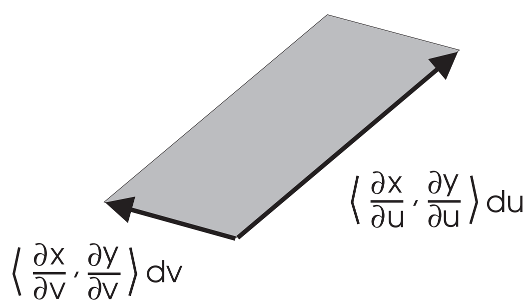
2. Use the transformation equations to substitute u and v for x and y.

3. Compute the absolute value of the Jacobian $|JT|$ and insert it in the integral.

Steps 1 and 2 seem reasonable; what about step 3? The idea is that for a small region in the u-v plane, the picture looks like this:



For small increments of u and v, T is well approximated by the derivative dT. Apply dT to the sides $(du, 0)$, $(0, dv)$ of the small rectangle in the u-v plane to get a small parallelogram in the x-y plane:



$$\begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix} \begin{bmatrix} du \\ 0 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial u} \\ \dfrac{\partial y}{\partial u} \end{bmatrix} du, \qquad \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix} \begin{bmatrix} 0 \\ dv \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial v} \end{bmatrix} dv.$$

Thus, $\left(\dfrac{\partial x}{\partial u}, \dfrac{\partial y}{\partial u}\right) du$ and $\left(\dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial v}\right) dv$ are the adjacent sides of the parallelogram. The parallelogram's area is

$$\left\| \begin{matrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{matrix} \right\| du\,dv = JT\,du\,dv.$$

That is, the Jacobian represents the factor by which small areas are scaled up by the transformation. This explains its presence in the change of variables formula.

Example. Compute the Jacobian of the transformation

$$(x,y) = T(u,v) = (5u^2 - v^2, u^3 + 2v^3).$$

$$JT = \det DT = \det \begin{bmatrix} 10u & -2v \\ 3u^2 & 6v^2 \end{bmatrix} = 60uv^2 + 6u^2v. \quad \square$$

Example. Compute the Jacobian of the transformation

$$(x,y,z) = T(u,v,w) = (5u + 3v + 7w, u^2 + v^2, u^2 - v^2).$$

$$JT = \det DT = \det \begin{bmatrix} 5 & 3 & 7 \\ 2u & 2v & 0 \\ 2u & -2v & 0 \end{bmatrix} = 7 \det \begin{bmatrix} 2u & 2v \\ 2u & -2v \end{bmatrix} = -28uv. \quad \square$$

Example. (a) Find a transformation $(x,y) = (f(u,v), g(u,v))$ which takes the unit square

$$\left\{ \begin{matrix} 0 \le u \le 1 \\ 0 \le v \le 1 \end{matrix} \right\} \quad \text{to the parallelogram with vertices} \quad A(2,-1),\ B(5,3),\ C(7,-2),\ D(4,-6).$$

(The vertices are listed counterclockwise around the parallelogram.)

(b) Let R be the parallelogram with vertices $A(2,-1)$, $B(5,3)$, $C(7,-2)$, $D(4,-6)$. (The vertices are listed counterclockwise around the parallelogram.)

Use the transformation in (a) and change of variables to compute

$$\iint_R (4x + 2y)\, dx\, dy.$$

(a) $\overrightarrow{AB} = (3,4)$ and $\overrightarrow{AC} = (2,-5)$, so the transformation is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

In component form, this is

$$x = 3u + 2v + 2, \quad y = 4u - 5v - 1. \quad \square$$

+

(b) The integrand is

$$4x + 2y = 4(3u + 2v + 2) + 2(4u - 5v - 1) = 20u - 2v + 6.$$

The Jacobian is

$$\left| \det \begin{bmatrix} 3 & 2 \\ 4 & -5 \end{bmatrix} \right| = |-23| = 23.$$

Hence,

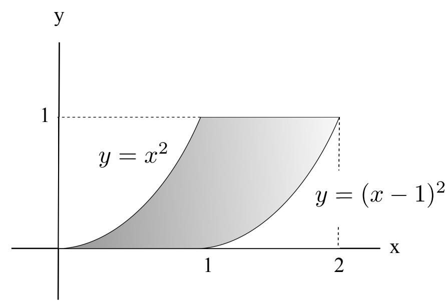
$$\iint_R (5x + 2y - 7)\, dx\, dy = \int_0^1 \int_0^1 (20u - 2v + 6) \cdot 23\, du\, dv = 23 \int_0^1 \left[10u^2 - 2uv + 6u\right]_0^1 dv =$$

$$23\int_0^1 (16-2v)\,dv = 23\left[16v - v^2\right]_0^1 = 345. \quad \square$$

Example. (a) Let R be the region in the x-y plane bounded on the left by $y = x^2$, $0 \le x \le 1$, on the right by $y = (x-1)^2$, $1 \le x \le 2$, above by $y = 1$, and below by $y = 0$. Use horizontal segments to find a coordinate transformation which carries the unit square $0 \le u \le 1$, $0 \le v \le 1$ to R.

(b) Compute $\displaystyle \iint_R (x^2 + 3y)\,dx\,dy$.

(a) Here is the region:



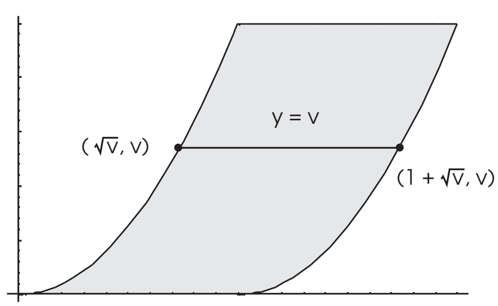
Notice that it can be broken up into horizontal segments. I will parametrize the segment at height $y = v$.

Recall that if P and Q are points, the line segment from P to Q can be parametrized by

$$(x, y) = (1-u)\cdot P + u \cdot Q, \quad 0 \le u \le 1.$$

You can derive this formula by finding the vector equation of the line through P and Q. However, it's easier to simply note that it has the right form for a line equation, and it passes through P (set $u = 0$) and Q (set $u = 1$) — so it must be the right thing.

The left hand curve is $y = x^2$. Set $y = v$: Then $x^2 = v$, so I get $x = \sqrt{v}$ — I want the positive square root, since the region is in the first quadrant. The left hand intersection point is $(\sqrt{v}, v)$.



The right hand curve is $y = (x-1)^2$. Set $y = v$: Then $(x-1)^2 = v$, so I get $x = 1 + \sqrt{v}$. The right hand intersection point is $(1 + \sqrt{v}, v)$.

Now I construct the segment joining the intersection points. By the discussion above, it is

$$(x, y) = (1-u)(\sqrt{v}, v) + u(1 + \sqrt{v}, v) = (\sqrt{v} - u\sqrt{v}, v - uv) + (u + u\sqrt{v}, uv) = (u + \sqrt{v}, v).$$

What are the limits? For the segment parametrization I'm using, u always goes from 0 to 1. v is the height of the segment, and the region extends from 0 to 1.

Thus, the coordinate transformation is

$$x = u + \sqrt{v}, \quad y = v, \quad 0 \le u \le 1, 0 \le v \le 1. \quad \square$$

(b) We use the transformation from (a):

$$x = u + \sqrt{v}, \quad y = v, \quad 0 \le u \le 1, 0 \le v \le 1.$$

The Jacobian is

$$J = \det \begin{bmatrix} 1 & \dfrac{1}{2\sqrt{v}} \\ 0 & 1 \end{bmatrix} = 1.$$

The integrand is

$$x^2 + 3y = (u + \sqrt{v})^2 + 3v = u^2 + 2u\sqrt{v} + 4v.$$

Hence,

$$\iint_R (x^2 + 3y)\, dx\, dy = \int_0^1 \int_0^1 (u^2 + 2u\sqrt{v} + 4v) \cdot 1\, du\, dv = \int_0^1 \left[\frac{1}{3}u^3 + u^2\sqrt{v} + 4uv\right]_0^1 dv =$$

$$\int_0^1 \left(\frac{1}{3} + \sqrt{v} + 4v\right) dv = \left[\frac{1}{3}v + \frac{2}{3}v^{3/2} + 2v^2\right]_0^1 = 3. \quad \square$$

Example. (a) Consider the coordinate transformation

$$(x, y) = f(u, v) = (u^2 - v^2, u - v).$$

Find the image R in the x-y plane of the square $0 \le u \le 1$, $0 \le v \le 1$.

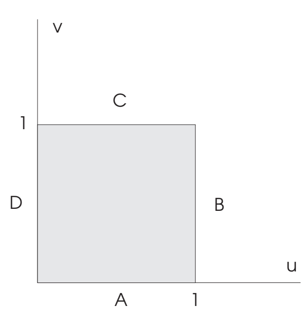
(b) Is the transformation injective (one-to-one) on the square?

(c) Use the change of variables formula to compute $\displaystyle\iint_R (x + y)\, dx\, dy$. Check by computing the integral directly. Does it work?

(a) In component form the transformation is

$$x = u^2 - v^2, \quad y = u - v.$$

A reasonable way to find the image is to look at where the four sides of the square go.



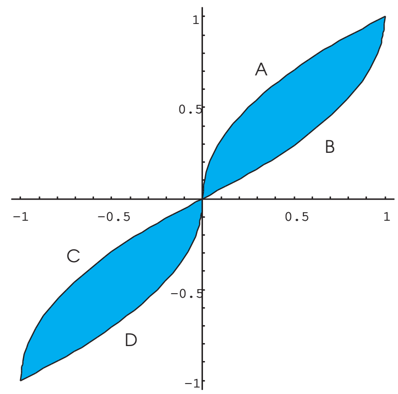
Look at side A: $0 \le u \le 1$, $v = 0$. The equations become $x = u^2$, $y = u$. Eliminating u, I get $x = y^2$. So this side of the square is mapped to the parabola $x = y^2$. Since $0 \le u \le 1$ and $y = u$, it is the part of the parabola which goes from $(0, 0)$ to $(1, 1)$.

Look at side B: $u = 1$, $0 \le v \le 1$. The equations become $x = 1 - v^2$, $y = 1 - v$. Then $v = 1 - y$, so $x = 1 - (1 - y)^2 = 2y - y^2$. This is a parabola; since $0 \le v \le 1$ and $y = 1 - v$, I have $0 \le y \le 1$. Hence, I want the part of the parabola which goes from $(0, 0)$ to $(1, 1)$.

Look at side C: $v = 1$, $0 \le u \le 1$. The equations become $x = u^2 - 1$, $y = u - 1$. Now $u = y + 1$, so $x = (y+1)^2 - 1 = y^2 + 2y$. This is a parabola again; since $0 \le u \le 1$ and $y = u - 1$, I want the part of the parabola which satisfies $-1 \le y \le 0$. It extends from $(0, 0)$ to $(-1, -1)$.

Finally, look at side D: $u = 0$, $0 \le v \le 1$. The equations become $x = -v^2$, $y = -v$. Then $x = -y^2$, which is another parabola. Since $0 \le v \le 1$ and $y = -v$, I want the part of the parabola for which $-1 \le y \le 0$. It extends from $(0, 0)$ to $(-1, -1)$.

The region R looks like two petals or leaves. I've drawn them below, with their edges labelled according to the sides of the square that mapped to them.



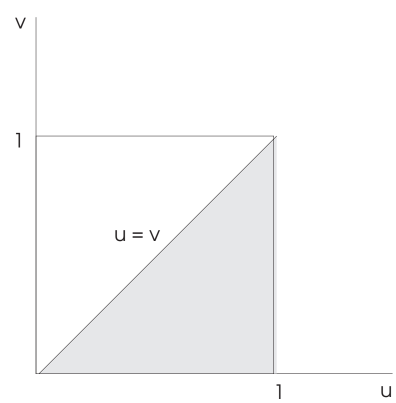
$\blacksquare$

(b) Do different imputs always produce different outputs? No — if $u = v$, then $x = 0$ and $y = 0$. That is, everything on the diagonal line $u = v$ is mapped to the origin in the x-y plane. It isn't one-to-one. $\blacksquare$

(c) The Jacobian is

$$J = \left|\det \begin{bmatrix} 2u & -2v \\ 1 & -1 \end{bmatrix}\right| = |-2(u-v)|.$$

There are two cases. If $u - v \ge 0$, then $J = |-2(u-v)| = 2(u-v)$. If $u - v \le 0$, then $J = |-2(u-v)| = -2(u-v)$. These two cases correspond to the halves of the square above and below the line $u = v$:



The bottom half (grey) is

$$0 \le u \le 1, \quad 0 \le v \le u \qquad (J = 2(u - v)).$$

The top half (white) is

$$0 \le u \le 1, \quad u \le v \le 1 \qquad (J = -2(u - v)).$$

The integrand is

$$x + y = u^2 - v^2 + u - v.$$

Therefore, the change of variables formula gives

$$\iint_R (x+y)\,dx\,dy \overset{?}{=} \int_0^1 \int_0^u (u^2-v^2+u-v)\cdot(2(u-v))\;dv\,du + \int_0^1 \int_u^1 (u^2-v^2+u-v)\cdot(-2(u-v))\;dv\,du = 0.$$

Now I'll check by doing the integral as is, in x-y coordinates. The following inequalities describe the two petals:

$$\left\{\begin{array}{c} 0 \le y \le 1 \\ y^2 \le x \le 2y - y^2 \end{array}\right\} \qquad \left\{\begin{array}{c} -1 \le y \le 0 \\ 2y + y^2 \le x \le -y^2 \end{array}\right\}$$

So the integral is

$$\int_0^1 \int_{y^2}^{2y-y^2} (x+y)\,dx\,dy + \int_{-1}^0 \int_{2y+y^2}^{-y^2} (x+y)\,dx\,dy = 0.$$

The transformation is not one-to-one on the line $u = v$, and the Jacobian vanishes along that line as well, so I'm not *entitled* to assume it will work — but it does. □