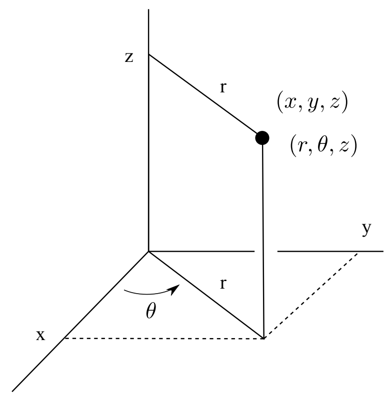
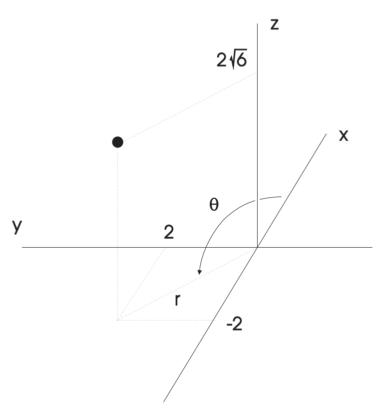
4-7-2018

Cylindrical Coordinates

Cylindrical coordinates assigned an ordered triple (r, θ, z) to points in space. If the rectangular coordinates of the point are (x, y, z), then (r, θ) are the polar coordinates of the point (x, y); the third coordinate z is the same in both cylindrical and rectangular.



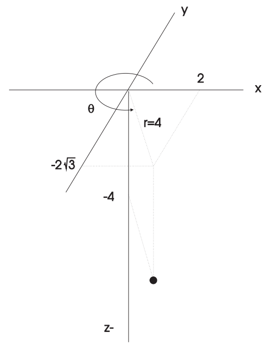
Example. A point has rectangular coordinates $(-2, 2, 2\sqrt{6})$. Find its cylindrical coordinates.



Since the point lies above the second quadrant, I've rotated the coordinates axes so that the second quadrant is "in front". From the picture, it's clear that $\theta = \dfrac{\pi}{2} + \dfrac{\pi}{4} = \dfrac{3\pi}{4}$. By Pythagoras, $r = 2\sqrt{2}$.

The cylindrical coordinates are $\left(2\sqrt{2}, \dfrac{3\pi}{4}, 2\sqrt{6}\right)$. □

Example. A point has cylindrical coordinates $\left(4, \dfrac{5\pi}{3}, -4\right)$. Find its rectangular coordinates.



The point lies below the fourth quadrant. Its rectangular coordinates are $(2, -2\sqrt{3}, -4)$. □

Example. Find the equation of the unit sphere $x^2 + y^2 + z^2 = 1$ in cylindrical coordinates.

Setting $x^2 + y^2 = r^2$, I get

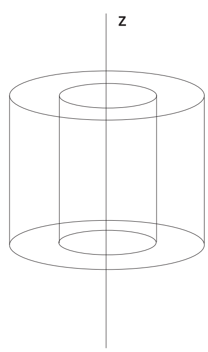
$$r^2 + z^2 = 1. \quad \square$$

Example. Convert $y = 2x$ to cylindrical coordinates.

$$\frac{y}{x} = 2, \quad \text{so} \quad \tan\theta = 2. \quad \square$$

Example. What is the set of points which satisfy the cylindrical coordinate equation $r^2 - 3r + 2 = 0$?

The equation is $(r - 1)(r - 2) = 0$, so $r = 1$ or $r = 2$. The locus consists of two concentric cylinders of radius 1 and 2 having the z-axis as their axis.



□

To convert a triple integral $\iiint_R f(x, y, z)\, dx\, dy\, dz$ to cylindrical coordinates:

1. Convert $f(x, y, z)$ to cylindrical coordinates using the polar coordinate conversion equations.

2. To obtain the limits of integration, describe the region R by inequalities in cylindrical coordinates.

3. Replace $dx\, dy\, dz$ with $r\, dr\, d\theta\, dz$.

In converting a double integral from rectangular to polar, we replace $dx\,dy$ with $r\,dr\,d\theta$, so this is reasonable from that point of view. You can see the factor of r from the change of variables formula. The transformation is

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

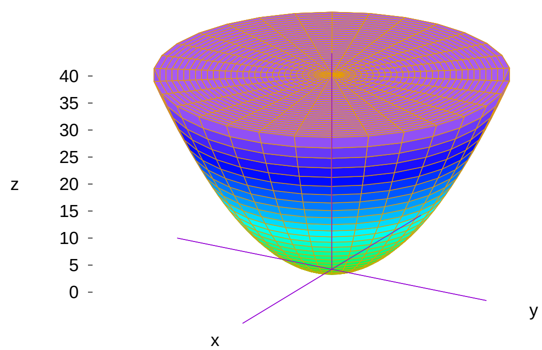
The Jacobian is

$$\det \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 01 & \end{bmatrix} = r(\cos\theta)^2 + r(\sin\theta)^2 = r.$$

Assuming that r is positive, the change of variables formula tells us to replace $dx\,dy\,dz$ with $r\,dr\,d\theta\,dz$.

Example. Let R be the region bounded below by $z = x^2 + y^2$ and above by $z = 36$. Compute

$$\iiint_R (x^2 + y^2 + 1)\,dx\,dy\,dz.$$



$\square$

The intersection of $z = x^2 + y^2$ and $z = 36$ is $x^2 + y^2 = 36$, a circle of radius 6 centered at the origin. Hence, the region of integration is

$$\left\{ \begin{matrix} 0 \le \theta \le 2\pi \\ 0 \le r \le 6 \\ r^2 \le z \le 36 \end{matrix} \right\}$$

Note that $x^2 + y^2 + 1 = r^2 + 1$. The integral is

$$\int_0^{2\pi} \int_0^6 \int_{r^2}^{36} (r^2 + 1) \cdot r\,dz\,dr\,d\theta = 2\pi \int_0^6 (r^3 + r)\,[z]_{r^2}^{36}\,dr = 2\pi \int_0^6 (r^3 + r)(36 - r^2)\,dr =$$

$$2\pi \int_0^6 (36r + 35r^3 - r^5)\,dr = 2\pi \left[18r^2 + \frac{35}{4}r^4 - \frac{1}{6}r^6\right]_0^6 = 8424\pi = 26464.77651\ldots. \quad \square$$
