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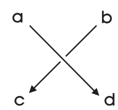
Determinants

The **determinant** of an square matrix is a number which is computed from the entries of the matrix. If A is a square matrix, the determinant of A is denoted by $\det A$ or $|A|$.

For 2×2 matrices, the formula is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Think of the following picture:



Example. Compute the determinant of $\begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}$.

$$\begin{vmatrix} 3 & -2 \\ 5 & 4 \end{vmatrix} = (3)(4) - (5)(-2) = 12 - (-10) = 22. \quad \square$$

For matrices which are 3×3 or larger, you can compute the determinant using a *recursive* algorithm called **expansion by cofactors**.

Example. Compute the determinant of

$$\begin{bmatrix} 1 & -1 & 3 \\ 4 & 5 & 2 \\ 1 & 0 & 7 \end{bmatrix}.$$

First, pick any row or column. *It's usually good to pick a row or column with lots of zeros.* I'll use column 2.

Go down column 2 one element at a time. For each element:

1. Cross out the row and column containing the element to leave a 2×2 matrix.

2. Find the product of the element, the determinant of the 2×2 matrix, and a plus or minus sign. The sign is determined by a "checkboard pattern":

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

The determinant is the sum of the products.

I'll work through the steps one element at a time. Cross out the row and column containing -1:

$$\begin{vmatrix} \cdot & \mathbf{-1} & \cdot \\ 4 & \cdot & 2 \\ 1 & \cdot & 7 \end{vmatrix}$$

Compute the 2×2 determinant:

$$\begin{vmatrix} 4 & 2 \\ 1 & 7 \end{vmatrix} = 26.$$

Multiply the element, the 2×2 determinant, and a minus sign:

$$(-1) \cdot 26 \cdot (-1) = 26.$$

Cross out the row and column containing 5:

$$\begin{vmatrix} 1 & \cdot & 3 \\ \cdot & \mathbf{5} & \cdot \\ 1 & \cdot & 7 \end{vmatrix}$$

Compute the 2×2 determinant:

$$\begin{vmatrix} 1 & 3 \\ 1 & 7 \end{vmatrix} = 4.$$

Multiply the element, the 2×2 determinant, and a plus sign:

$$5 \cdot 4 \cdot (+1) = 20.$$

Cross out the row and column containing 0:

$$\begin{vmatrix} 1 & \cdot & 3 \\ 4 & \cdot & 2 \\ \cdot & \mathbf{0} & \cdot \end{vmatrix}$$

Compute the 2×2 determinant:

$$\begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = -10.$$

Multiply the element, the 2×2 determinant, and a minus sign:

$$0 \cdot (-10) \cdot (-1) = 0.$$

The total is $26 + 20 + 0 = 46$. □

As an exericse, try expanding the determinant of this matrix using another row or column and see if you get the same answer.

Example. Compute $\left| \begin{matrix} -3 & 1 & 1 \\ 1 & 2 & 1 \\ 5 & -1 & 3 \end{matrix} \right]$.

I'll expand by cofactors of the first row:

$$\begin{vmatrix} -3 & 1 & 1 \\ 1 & 2 & 1 \\ 5 & -1 & 3 \end{vmatrix} = (-3) \left| \begin{matrix} 2 & 1 \\ -1 & 3 \end{matrix} \right] - (1) \begin{vmatrix} 1 & 1 \\ 5 & 3 \end{vmatrix} + (1) \begin{vmatrix} 1 & 2 \\ 5 & -1 \end{vmatrix} =$$

$$(-3)(7) - (1)(-2) + (1)(-11) = -30. \quad \square$$

Notice that expansion by cofactors reduces the computation of an $n \times n$ determinant to the computation of n $(n-1) \times (n-1)$ determinants.

There are other approaches you can use to compute determinants. For instance, in linear algebra you learn about **row reduction**, which provides a fairly efficient way to compute determinants for larger matrices.

Here are some properties of determinants.

Proposition. (a) If A has two equal rows, then $\det A = 0$.

(b) If two rows of A are swapped, the value of the determinant is multiplied by -1:

$$\det \begin{bmatrix} \vdots \\ \leftarrow \vec{x} \rightarrow \\ \vdots \\ \leftarrow \vec{y} \rightarrow \\ \vdots \end{bmatrix} = -\det \begin{bmatrix} \vdots \\ \leftarrow \vec{y} \rightarrow \\ \vdots \\ \leftarrow \vec{x} \rightarrow \\ \vdots \end{bmatrix}.$$

(c) The determinant of a sum is the sum of the determinants *one row at a time*:

$$\det \begin{bmatrix} \text{(FOO)} \\ \leftarrow \vec{x} + \vec{y} \rightarrow \\ \text{(BAR)} \end{bmatrix} = \det \begin{bmatrix} \text{(FOO)} \\ \leftarrow \vec{x} \rightarrow \\ \text{(BAR)} \end{bmatrix} + \det \begin{bmatrix} \text{(FOO)} \\ \leftarrow \vec{y} \rightarrow \\ \text{(BAR)} \end{bmatrix}.$$

(The parts of the matrices labelled “FOO” and “BAR” are *the same* in all 3 matrices: They don't change. The sum occurs in a single row.)

(d) A number may be factored out of *one row of a determinant at a time*:

$$\det \begin{bmatrix} \text{(FOO)} \\ \leftarrow k \cdot \vec{x} \rightarrow \\ \text{(BAR)} \end{bmatrix} = k \cdot \det \begin{bmatrix} \text{(FOO)} \\ \leftarrow \vec{x} \rightarrow \\ \text{(BAR)} \end{bmatrix}.$$

(e) The determinant of a product is the product of the determinants:

$$\det(AB) = (\det A)(\det B).$$

(f) The determinant of the $n \times n$ identity matrix is 1:

$$\det \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = 1.$$

Proof. The proofs that these properties hold for arbitrary $n \times n$ matrices are fairly involved; you'd see them in a course in linear algebra.

I'll verify that a couple of the properties hold in some special cases.

As an example of (a), here's the determinant of a 3×3 matrix with two equal rows, which I'm computing by expanding by cofactors of row 1:

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ d & e & f \end{bmatrix} = a \cdot \det \begin{bmatrix} e & f \\ e & f \end{bmatrix} - b \cdot \det \begin{bmatrix} d & f \\ d & f \end{bmatrix} + c \cdot \det \begin{bmatrix} d & e \\ d & e \end{bmatrix} =$$

$$a(ef - fe) - b(df - fd) + c(de - ed) = 0.$$

Here's an example of (e) with 2×2 matrices,

$$\begin{bmatrix} x + x' & y + y' \\ a & b \end{bmatrix} = (x + x') \cdot b - (y + y') \cdot a = (xb - ya) + x'b - y'a) = \det \begin{bmatrix} x & y \\ a & b \end{bmatrix} + \det \begin{bmatrix} x' & y' \\ a & b \end{bmatrix}. \quad \square$$

Some of these properties are illustrated in the following examples.

Example. Suppose that

$$\left|\begin{matrix} a & b \\ c & d \end{matrix}\right| = 10.$$

Compute:

(a) $\left|\begin{matrix} c & d \\ a & b \end{matrix}\right|$.

(b) $\left|\begin{matrix} 3a & 3b \\ 2c & 2d \end{matrix}\right|$.

(c) $\left|\begin{matrix} 4a+c & 4b+d \\ c & d \end{matrix}\right|$.

(a) Swapping two rows multiplies the determinant by -1:

$$\left|\begin{matrix} c & d \\ a & b \end{matrix}\right| = -\left|\begin{matrix} a & b \\ c & d \end{matrix}\right| = -10. \quad \square$$

(b) Factor 3 out of the first row, then factor 2 out of the second row:

$$\left|\begin{matrix} 3a & 3b \\ 2c & 2d \end{matrix}\right| = 3 \cdot \left|\begin{matrix} a & b \\ 2c & 2d \end{matrix}\right| = 2 \cdot 3 \left|\begin{matrix} a & b \\ c & d \end{matrix}\right| = 6 \cdot 10 = 60. \quad \square$$

(c) Break up the determinant using a sum of $(4a, 4b)$ and (c, d) in the first row. Factor 4 out of the first determinant; the second determinant is 0 because the matrix has two equal rows.

$$\left|\begin{matrix} 4a+c & 4b+d \\ c & d \end{matrix}\right| = \left|\begin{matrix} 4a & 4b \\ c & d \end{matrix}\right| + \left|\begin{matrix} c & d \\ c & d \end{matrix}\right| = 4 \cdot \left|\begin{matrix} a & b \\ c & d \end{matrix}\right| + 0 = 4 \cdot 10 = 40. \quad \square$$

Example. Give specific 2×2 matrices A and B for which

$$\det(A+B) \neq \det A + \det B.$$

$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right) = \det\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0.$$

$$\det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \det\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = 1 + 1 = 2. \quad \square$$
