4-1-2018

Double Integrals in Polar

It's often useful to change variables and convert a double integral from rectangular coordinates to polar coordinates. Suppose you're trying to convert the following integral to polar coordinates:

$$\iint_D f(x,y)\,dx\,dy.$$

1. Convert the function $f(x,y)$ to polar by using the polar-rectangular conversion equations:

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x},$$

$$x = r\cos\theta, \quad y = r\sin\theta.$$

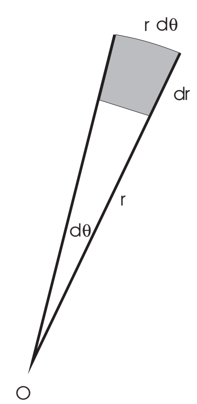
2. Replace $dx\,dy$ with $r\,dr\,d\theta$.

3. Describe the region of integration D by inequalities in polar and use the inequalities to change the limits.

The only thing which requires explanation is why you replace $dx\,dy$ with $r\,dr\,d\theta$. One way to understand this is to use the **change-of-variables** formula for double integrals. This says that

$$dx\,dy = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} dr\,d\theta = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} dr\,d\theta = \left[r(\cos\theta)^2 + r(\sin\theta)^2\right]\,dr\,d\theta = r\,dr\,d\theta.$$

Heuristically, you can picture this by considering a small wedge of area in the polar grad:



The "box" has height dr and width $r\,d\theta$ — the width coming from the formula for an arc of radius r subtended by an angle $d\theta$. The area of the box should be $r\,dr\,d\theta$.

Example. Convert $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{(x^2+y^2+1)^{3/2}}\,dx\,dy$ to polar and compute the integral.

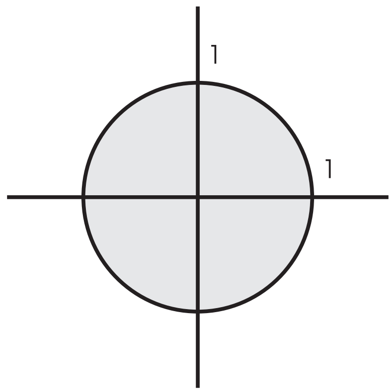
This integral would be horrible to compute in rectangular coordinates. In polar, it's pretty easy. First, convert the function:

$$\frac{1}{(x^2+y^2+1)^{3/2}} = \frac{1}{(r^2+1)^{3/2}}.$$

I'll replace $dx\,dy$ with $r\,dr\,d\theta$ when I set up the integral.
To convert the limits, pull the original limits off as inequalities:

$$\left\{\begin{array}{c} -1 \le x \le 1 \\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2} \end{array}\right\}$$

Draw the region described by the inequalities. It is the interior of the circle $x^2 + y^2 = 1$:



Describe the region by inequalities in polar:

$$\left\{\begin{array}{c} 0 \le \theta \le 2\pi \\ 0 \le r \le 1 \end{array}\right\}$$

Put the inequalities on the integral and compute:

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{(x^2+y^2+1)^{3/2}}\,dx\,dy = \int_0^{2\pi}\int_0^1 \frac{1}{(r^2+1)^{3/2}} \cdot r\,dr\,d\theta = \int_0^{2\pi} \left[-\frac{1}{\sqrt{r^2+1}}\right]_0^1 d\theta =$$

$$\int_0^{2\pi} \frac{1}{2}\,d\theta = \left[\frac{1}{2}\theta\right]_0^{2\pi} = \pi.$$

(I did $\displaystyle\int \frac{r\,dr}{(r^2+1)^{3/2}}$ by using the substitution $u = r^2 + 1$.) □

Here is a rule of thumb that was evident in the last problem:

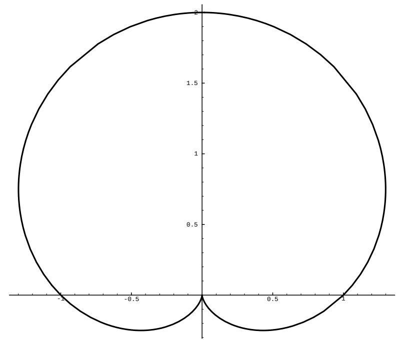
Think about converting to polar when the double integral contains terms of the form $x^2 + y^2$.

You can use double integrals in polar to compute areas of regions in the x-y-plane. Just as with x-y double integrals,

$$\iint_D r\,dr\,d\theta \quad \text{gives the area of } D.$$

However, you can often use a single integral to compute the area — the double integral is superfluous. For this reason, the next example isn't particularly practical; it just illustrates the idea.

Example. Use a double integral to compute the area of the region inside the cardioid $r = 1 + \sin\theta$.



I know the cardioid is traced out once as θ goes from 0 to 2π, so the region inside is described by the inequalities

$$\left\{\begin{matrix} 0 \le \theta \le 2\pi \\ 0 \le r \le 1 + \sin\theta \end{matrix}\right\}$$

The area is given by the double integral

$$\int_0^{2\pi}\int_0^{1+\sin\theta} r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{1}{2}r^2\right]_0^{1+\sin\theta} d\theta = \int_0^{2\pi}\frac{1}{2}(1+\sin\theta)^2\,d\theta = \left[\frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = 3\pi.$$

(I did the θ integral by multiplying $(1+\sin\theta)^2$ out, then applying the double angle formula to $(\sin\theta)^2$.)

Do you notice what happened in the third step? I got the same integral $\int_0^{2\pi}\frac{1}{2}(1+\sin\theta)^2\,d\theta$ that I would have gotten using the old single-variable formula

$$\int_a^b \frac{1}{2}r^2\,d\theta.$$

It wasn't necessary to use a double integral to find this area. □

Example. Compute $\int_{-\infty}^{\infty} e^{-x^2}\,dx$.

This single variable integral is important in probability. Here's the trick to computing it: Let

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx.$$

The variable in a definite integral is a *dummy variable* — the value of the integral isn't changed if I change the letter. So

$$I = \int_{-\infty}^{\infty} e^{-y^2}\,dy.$$

Multiply the two equations:

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy.$$

Convert to polar: $e^{-(x^2+y^2)} = e^{-r^2}$, and $dx\,dy$ will be replaced with $r\,dr\,d\theta$. The region is

$$\left\{\begin{matrix} -\infty < x < +\infty \\ -\infty < y < +\infty \end{matrix}\right\}$$

This is the whole x-y plane! In polar, this is

$$\left\{\begin{array}{c} 0 \le \theta \le 2\pi \\ 0 \le r < +\infty \end{array}\right\}$$

So

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\cdot r\,dr\,d\theta = \int_0^{2\pi}\left(\lim_{c\to\infty}\int_0^c re^{-r^2}\,dr\right)d\theta =$$

$$\int_0^{2\pi}\left(\lim_{c\to\infty}\left[-\frac{1}{2}e^{-r^2}\right]_0^c\right)d\theta = \frac{1}{2}\int_0^{2\pi}\left(\lim_{c\to\infty}(1-e^{-c^2})\right)d\theta = \frac{1}{2}\int_0^{2\pi} d\theta = \pi.$$

(I did $\int re^{-r^2}\,dr$ using the substitution $u = -r^2$.)
Therefore, $I = \sqrt{\pi}$ — that is,

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}. \quad \Box$$

Example. Compute the integral by converting to polar coordinates:

$$\int_0^2\int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dx\,dy$$

$\sqrt{x^2+y^2} = \sqrt{r^2} = r$, and I'll replace $dx\,dy$ with $r\,dr\,d\theta$.
Pull off the limits of integration:

$$\left\{\begin{array}{c} 0 \le x \le 2 \\ 0 \le y \le \sqrt{2x-x^2} \end{array}\right\}$$

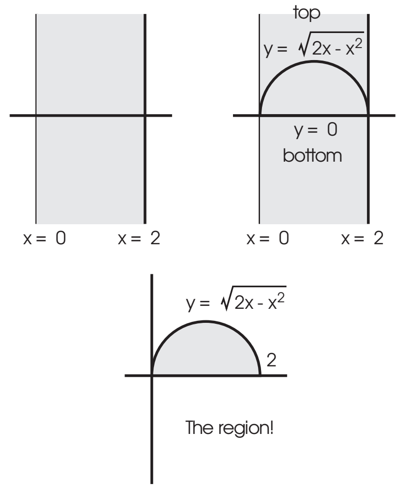
Draw the region described by the inequalities. *Do the "number" inequalities first.* $0 \le x \le 2$ tells you the region is between the vertical lines $x = 0$ and $x = 2$.

The y-inequalities $0 \le y \le \sqrt{2x-x^2}$ tell you that the top curve for the region is $y = \sqrt{2x-x^2}$ and the bottom curve is $y = 0$ — the same kind of thing you do when you used (single) integrals to compute the area between curves.

To recognize $y = \sqrt{2x-x^2}$, complete the square:

$$\begin{aligned} y &= \sqrt{2x-x^2} \\ y^2 &= 2x-x^2 \\ x^2-2x+y^2 &= 0 \\ x^2-2x+1+y^2 &= 1 \\ (x-1)^2+y^2 &= 1 \end{aligned}$$

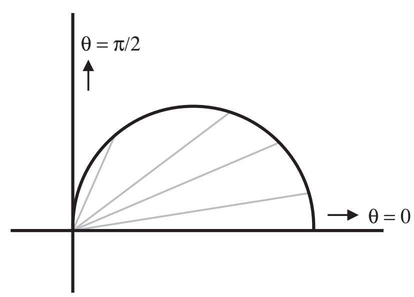
$y = \sqrt{2x - x^2}$ is the top half of a circle of radius 1 centered at $(1, 0)$. Here's the picture:



Now I describe the region in polar. Convert the circle to polar:

$$\begin{aligned} x^2 - 2x + y^2 &= 0 \\ x^2 + y^2 &= 2x \\ r^2 &= 2r\cos\theta \\ r &= 2\cos\theta \end{aligned}$$

The top half is traced out as θ goes from 0 to $\dfrac{\pi}{2}$ — think of a searchlight beam turning to trace out the curve:



Therefore, the polar inequalities are

$$\left\{ \begin{matrix} 0 \le \theta \le \dfrac{\pi}{2} \\ 0 \le r \le 2\cos\theta \end{matrix} \right\}$$

So

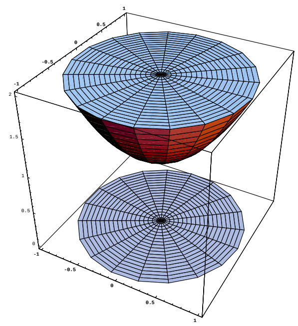
$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2 + y^2}\, dx\, dy = \int_0^{\pi/2} \int_0^{2\cos\theta} r \cdot r\, dr\, d\theta = \int_0^{\pi/2} \left[\frac{1}{3} r^3\right]_0^{2\cos\theta} d\theta = \frac{8}{3} \int_0^{\pi/2} (\cos\theta)^3\, d\theta =$$

$$\frac{8}{3} \int_0^{\pi/2} \left(1 - (\sin\theta)^2\right) (\cos\theta)\, d\theta = \frac{16}{9}.$$

(I did the θ integral with the substitution $u = \sin\theta$.) □

Example. Compute the volume of the region $x^2 + y^2 + 1 \leq z \leq 2$.

Here is the region:



The "bowl" is the surface $z = x^2 + y^2 + 1$.
The intersection of $z = 2$ and $z = x^2 + y^2 + 1$ is

$$2 = x^2 + y^2 + 1, \quad \text{or} \quad x^2 + y^2 = 1.$$

This is the curve where the bowl hits the plane, and you can see it's the unit circle (moved up to $z = 2$).
Hence, if you project the region down into the x-y plane, you'll get the interior of the circle $x^2 + y^2 = 1$. I'll convert to polar. The projection is

$$\left\{ \begin{matrix} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 1 \end{matrix} \right\}$$

To find the volume, I integrate top − bottom, which is

$$2 - (x^2 + y^2 + 1) = 1 - x^2 - y^2 = 1 - r^2.$$

Since I'm converting to polar, I replace $dx\,dy$ with $r\,dr\,d\theta$. The volume is

$$V = \int_0^{2\pi} \int_0^1 (1 - r^2) r\,dr\,d\theta = \int_0^{2\pi} \int_0^1 (r - r^3)\,dr\,d\theta = \int_0^{2\pi} \left[\frac{1}{2}r^2 - \frac{1}{4}r^4\right]_0^1 d\theta =$$

$$\frac{1}{4} \int_0^{2\pi} d\theta = \frac{\pi}{2}. \quad \square$$