3-26-2018

Double Integrals

A **double integral** is an integral

$$\iint_R f(x,y)\,dx\,dy.$$

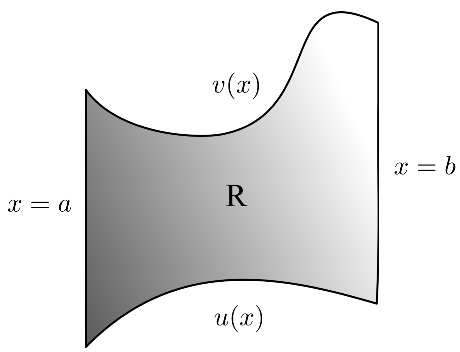
R is a **region** in $\mathbb{R}^2$, and $f(x,y)$ is an **integrable function**.

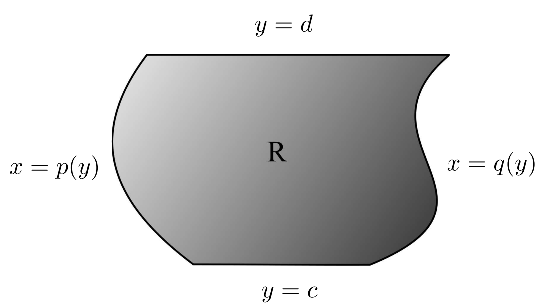
Under appropriate conditions — for example, if f is continuous function — you can compute a double integral as an **iterated integral**:

$$\iint_R f(x,y)\,dx\,dy = \int_a^b \left(\int_{u(x)}^{v(x)} f(x,y)\,dy\right) dx \quad \text{or} \quad \iint_R f(x,y)\,dx\,dy = \int_c^d \left(\int_{p(y)}^{q(y)} f(x,y)\,dx\right) dy.$$

In the first case, the region is described by inequalities

$$R = \left\{ \begin{array}{c} a \le x \le b \\ u(x) \le y \le v(x) \end{array} \right\}.$$





In the second case, the region is described by inequalities

$$R = \left\{ \begin{array}{c} c \le y \le d \\ p(y) \le x \le q(y) \end{array} \right\}.$$

Example. Compute

$$\iint_R (4x - 6y + 3)\,dx\,dy, \quad \text{where} \quad R = \left\{ \begin{array}{c} 0 \le x \le 1 \\ -1 \le y \le 1 \end{array} \right\}.$$

I may compute the double integral as an iterated integral, integrating with respect to one variable at a time while holding the other variable constant.

Since all the limits are numbers, I can integrate first with respect to x and then with respect to y, or the other way around. In this problem, there is no reason to prefer one order to another.

$$\int_{-1}^1 \int_0^1 (4x - 6y + 3)\,dx\,dy = \int_{-1}^1 \left[2x^2 - 6xy + 3x\right]_0^1 dy = \int_{-1}^1 (5 - 6y)\,dy = \left[5y - 3y^2\right]_{-1}^1 = 10.$$

Notice how the limits of integration are matched with the integration variable from inside out:

$$\int_{-1}^1 \int_0^1 (4x - 6y + 3)\,dx\,dy$$

Thus, you integrate with respect to x first (holding y constant), then with respect to y. You might try doing this integral in the other order, with respect to y and then x:

$$\int_0^1 \int_{-1}^1 (4x - 6y + 3)\, dy\, dx.$$

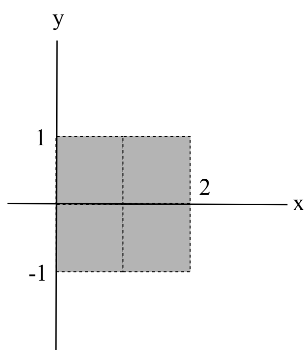
You should get the same answer. □

Example. Sketch the region of integration for the double integral:

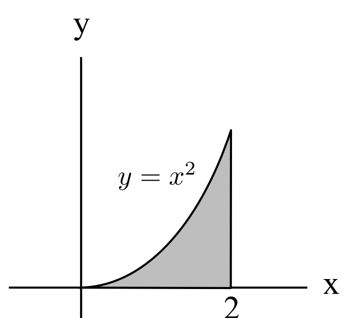
(a) $\displaystyle\int_{-1}^1 \int_0^2 f(x, y)\, dx\, dy.$

(b) $\displaystyle\int_0^2 \int_0^{x^2} f(x, y)\, dy\, dx.$

(a)



(b)



□

Example. Compute

$$\iint_R \cos e^x\, dx\, dy, \quad \text{where} \quad R = \left\{ \begin{array}{c} 0 \le x \le 1 \\ 0 \le y \le e^x \end{array} \right\}.$$

Note that since y has a variable limit, I must integrate with respect to y first, then x. When I integrate with respect to y, I hold x constant. Thus, the term "$\cos e^x$" is constant with respect to y when I do the first integration.

$$\int_0^1 \int_0^{e^x} \cos e^x\, dy\, dy = \int_0^1 \left[y \cos e^x\right]_0^{e^x}\, dx = \int_0^1 e^x \cos e^x\, dx = \int_1^e e^x \cos u \cdot \frac{du}{e^x} =$$

$$\left[u = e^x, \quad du = e^x\,dx, \quad dx = \frac{du}{e^x}; \quad x = 0, u = 1; x = 1, u = e\right]$$

$$\int_1^e \cos u\,du = [\sin u]_1^e = \sin e - \sin 1 = -0.43068\ldots. \quad \square$$

Example. Compute

$$\iint_R e^{(3y-y^3)}\,dx\,dy, \quad \text{where} \quad R = \left\{ \begin{array}{c} y^2 \le x \le 1 \\ -1 \le y \le 1 \end{array} \right\}.$$

$$\int_{-1}^1 \int_{y^2}^1 e^{(3y-y^3)}\,dx\,dy = \int_{-1}^1 \left[xe^{(3y-y^3)}\right]_{y^2}^1 dy = \int_{-1}^1 (1-y^2)e^{(3y-y^3)}\,dy = \int_?^? (1-y^2)e^u \cdot \frac{du}{3(1-y^2)} =$$

$$\left[u = 3y - y^3, \quad du = 3(1-y^2)\,dy, \quad dy = \frac{du}{3(1-y^2)}\right]$$

$$\frac{1}{3}\int_?^? e^u\,du = \frac{1}{3}[e^u]_?^? = \frac{1}{3}\left[e^{(3y-y^3)}\right]_{-1}^1 = \frac{1}{3}\left(e^2 - e^{-2}\right) = 2.41790\ldots. \quad \square$$

The double integral over a region R of the constant function 1 is just the area of R.

$$\iint_R 1\,dx\,dy = \text{area}(R).$$

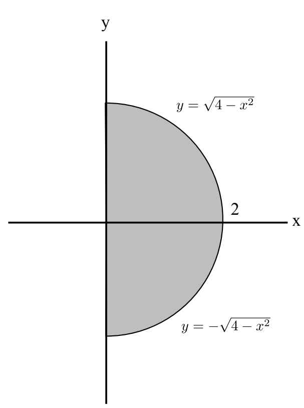
Example. Evaluate the integral without computing any antiderivatives:

$$\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\,dx.$$

The region is

$$\left\{ \begin{array}{c} 0 \le x \le 2 \\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2} \end{array} \right\}$$

Note that $y = \pm\sqrt{4-x^2}$ gives $x^2 + y^2 = 4$. But we're only looking at the part from $x = 0$ to $x = 2$.



The region is a half circle of radius 2, whose area is

$$\frac{1}{2}\pi \cdot 2^2 = 2\pi.$$

Hence,

$$\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\,dx = 2\pi. \quad \square$$

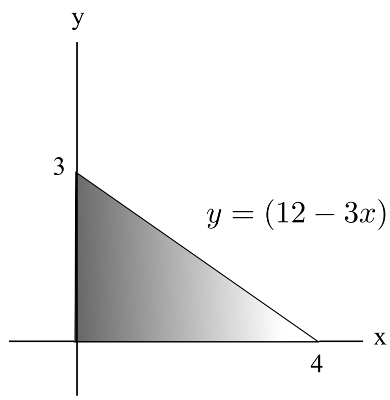
Example. Evaluate the integral without computing any antiderivatives:

$$\int_0^4 \int_0^{(12-3x)/4} dy\,dx.$$

The region is

$$\left\{ \begin{matrix} 0 \le x \le 4 \\ 0 \le y \le \dfrac{12-3x}{4} \end{matrix} \right\}$$

Now $y = \dfrac{12-3x}{4}$ gives $3x + 4y = 12$, a line with x-intercept 4 and y-intercept 3.



The region is a triangle in the first quadrant, whose area is

$$\int_0^4 \int_0^{(12-3x)/4} dy\,dx = \frac{1}{2} \cdot 4 \cdot 3 = 6. \quad \square$$

Multiple integrals satisfy the **monotonicity condition**: "Bigger functions give bigger integrals".

Proposition. Suppose f and g are integrable on a region R and $f(x) \ge g(x)$ for all $x \in R$. Then

$$\int_R f \ge \int_R g. \quad \square$$

Example. Suppose $f(x, y) \ge 6xy$ for all (x, y). Use this to obtain a lower bound for

$$\int_0^1 \int_0^x f(x, y)\,dy\,dx.$$

$$\int_0^1 \int_0^x f(x, y)\,dy\,dx \ge \int_0^1 \int_0^x 6xy\,dy\,dx = \int_0^1 \left[3xy^2\right]_0^x dx = \int_0^1 3x^3\,dx =$$

$$\left[\frac{3}{4}x^4\right]_0^1 = \frac{3}{4}.$$

Thus,

$$\int_0^1 \int_0^x f(x, y)\,dy\,dx \ge \frac{3}{4}. \quad \square$$
