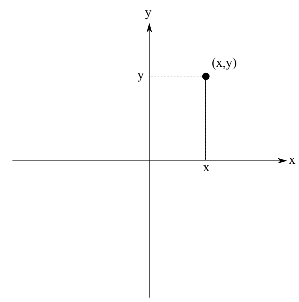
Euclidean Space

The **real numbers** are denoted by $\mathbb{R}$. I'll assume you're familiar with the basic properties of $\mathbb{R}$, but I'll mention less familiar things as they come up.

The standard 2-dimensional plane is denoted $\mathbb{R}^2$. It consists of ordered pairs of real numbers:

$$\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}.$$

You're probably familiar with how a point (x, y) is located in the plane:



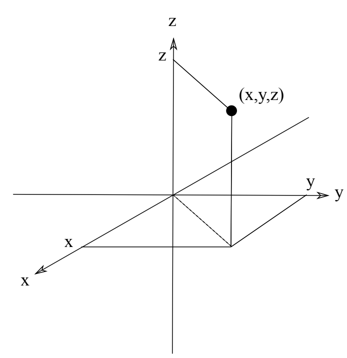
The "x" and "y" on the axes label the positive x-axis and positive y-axis.

Similarly, 3-dimensional space is denoted $\mathbb{R}^3$. It consists of ordered triples of real numbers:

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}.$$

(There are also $\mathbb{R}^4$, $\mathbb{R}^5$, and so on, defined in similar fashion.)

Just as it's conventional to use x and y to denote the first and second coordinate variables in $\mathbb{R}^2$, it is conventional to use x, y, and z to denote the first, second, and third coordinate variables in $\mathbb{R}^3$. The picture shows how a typical point (x, y, z) is located in space:



The "x", "y", and "z" on the axes label the positive x-axis, the positive y-axis, and the positive z-axis.

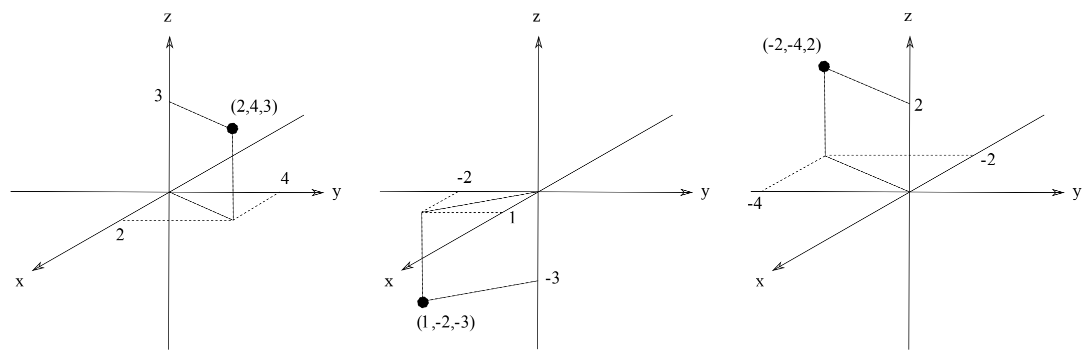
At this point, I should note a convention that we'll always follow. The labelling of the positive axes in $\mathbb{R}^2$ and $\mathbb{R}^3$ follow the "right-hand rule". In the x-y-plane, you curl the fingers of your *right* hand from the positive x-axis to the positive y-axis through a $90°$ angle. Check for yourself with the picture above.

(Note that in some computer graphics applications — for instance, in the SVG graphics language — it's conventional to have the positive y-axis going "downward" rather than "upward".)

For $\mathbb{R}^3$, curl the fingers of your *right* hand from the positive x-axis to the positive y-axis. As you do so, your thumb points in the direction of the positive z-axis. Check for yourself with the picture above.

Graphing things in 3 dimensions is obviously harder than graphing things in 2 dimensions. While you don't need to have great artistic skills, you should practice making diagrams in 3 dimensions as they are a huge aid to understanding. Short of taking a drawing class, the best approach might be to copy other peoples' pictures until you get the idea. For starters, you can practice plotting points in $\mathbb{R}^3$.

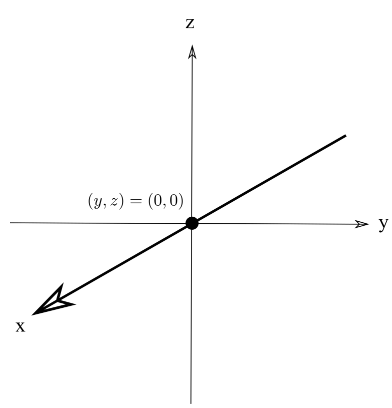
Example. Plot the points $(2, 4, 3)$, $(1, -2, -3)$, and $(-2, -4, 2)$. The pictures below aren't perfectly scaled; I just want to locate the points in approximately the right places.



□

The axes and the coordinate planes. Consider the x-axis. It is perpendicular to the y-z plane, and passes through the origin $(y, z) = (0, 0)$ of the y-z-plane. Therefore, it is determined by the equations

$$y = 0, \quad z = 0.$$



I'll discuss **lines** later on, and in particular their equations in parametric form. The parametric equations for the x-axis are

$$x = t, \quad y = 0, \quad z = 0.$$

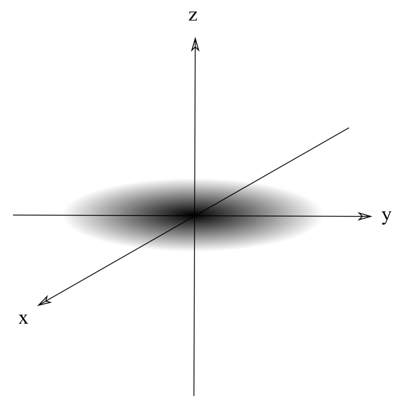
Similarly, the y-axis is

$$x = 0, \quad z = 0.$$

The z-axis is

$$x = 0, \quad y = 0.$$

Now consider the x-y plane. It consists of all the points at "z-level" zero — that is $z = 0$.

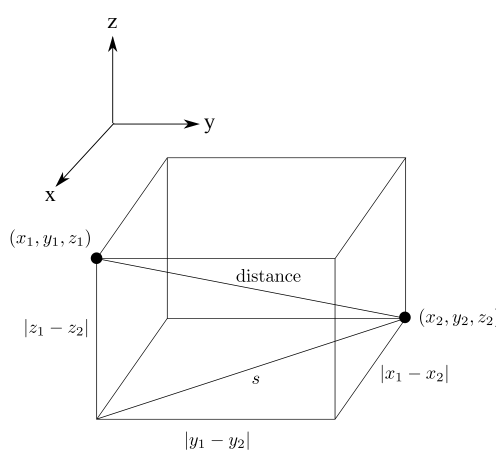


Likewise, the y-z plane is $x = 0$, and the x-z plane is $y = 0$.

Distance. The distance between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by

$$\text{distance} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

(You can write "$(x_2 - x_1)^2$", and so on, instead; the squaring makes the order of subtraction irrelevant.)
Here's where the formula comes from.



The box has sides of lengths $|x_1 - x_2|$, $|y_1 - y_2|$, and $|z_1 - z_2|$. By Pythagoras' Theorem, the diagonal s of the bottom of the box is

$$s = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

I can drop the absolute values because I'm squaring the terms.

Again by Pythagoras' Theorem, the distance is

$$\text{distance} = \sqrt{s^2 + (z_1 - z_2)^2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

Example. Find the distance between the points $(4, -1, 5)$ and $(2, 6, -3)$.

$$\sqrt{(4 - 2)^2 + (-1 - 6)^2 + (5 - (-3))^2} = \sqrt{4 + 49 + 64} = \sqrt{117}. \quad \square$$
