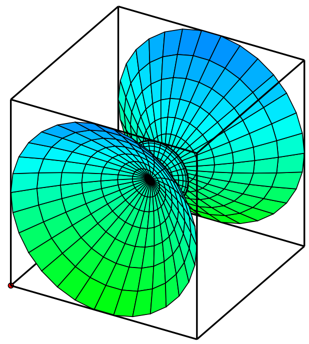
Functions of Several Variables

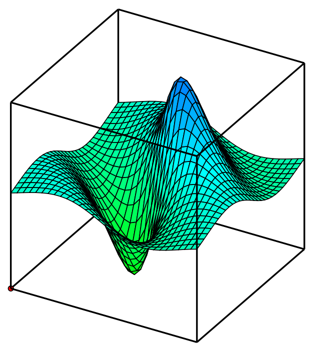
A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is a **function of several variables** if $n > 1$ — that is, if there is more than one input variable.

For example a function $f : \mathbb{R}^2 \to \mathbb{R}^3$ is a **parametrized surface** in $\mathbb{R}^3$. Here's a picture of

$$f(u, v) = ((u^2 - 1)\cos v, u^3, (u^2 - 1)\sin v).$$



Or consider a function $f : \mathbb{R}^2 \to \mathbb{R}$. Its **graph** is a surface in $\mathbb{R}^3$. Here's a picture of the graph of $f(x, y) = \dfrac{\sin(x + y)}{x^2 + y^2 + 1}$:



Functions of several variables occur in many real world situations — in fact, most measurable quantities depend on many factors or variables. For example, the temperature at a point in space may be a function of its coordinates (x, y, z). The ideal gas law $pV = nRT$ relates the pressure p, the volume V, and the temperature T of an ideal gas. I can regard any one of these three variables as a function of the other two; for example, writing $p = \dfrac{nRT}{V}$ views p as a function of V and T.

Example. A function $f : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$f(x, y, z) = (2x + y, x - z, x^2 y + z).$$

Evaluate $f(1, 4, -1)$, $f(2, 3, 0)$, and $f(6, 1, 1)$.

You substitute values into a function of several variables in the obvious way. For instance, to evaluate $f(1, 4, -1)$, I set $x = 1$, $y = 4$, and $z = -1$ in the formula for f:

$$f(1, 4, -1) = (2 \cdot 1 + 4, 1 - (-1), 1^2 \cdot 4 + (-1)) = (6, 2, 3).$$

Likewise,

$$f(2, 3, 0) = (7, 2, 12) \quad \text{and} \quad f(6, 1, 1) = (13, 5, 37). \quad \square$$

Definition. For a function $f : \mathbb{R}^n \to \mathbb{R}^m$:

(a) The **domain** is the set of all points in $\mathbb{R}^n$ where f is defined.

(b) The **image** (or the **range**) is the set of all outputs of f in $\mathbb{R}^m$.

Remark. I'm following the usual convention: For functions $\mathbb{R}^n \to \mathbb{R}^m$ to refer to the set of points in $\mathbb{R}^n$ where the function is defined as the **domain** of the function. Thus, for the function $f(x, y) = \dfrac{1}{x - y}$, the "domain" is the set of points (x, y) such that $x - y \neq 0$, i.e. $x \neq y$.

In more advanced courses, a more precise definition of a **function** requires that the "domain" be included as part of the function's definition. In that context, what we're calling the "domain" is referred to as the **natural domain** (the "biggest possible set" where the function is defined).

Example. A function of 2 variables is defined by

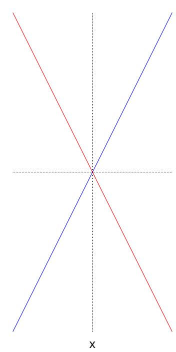
$$f(x, y) = \frac{x^2 + y^2 + 4}{y^2 - 4x^2}.$$

Describe the set of points (x, y) for which f is undefined. What is the **domain** of f?

$f(x, y) = \dfrac{x^2 + y^2 + 4}{y^2 - 4x^2}$ is undefined when the denominator is 0:

$$\begin{aligned} y^2 - 4x^2 &= 0 \\ (y - 2x)(y + 2x) &= 0 \end{aligned}$$

Thus, f is undefined for points (x, y) on either of the lines $y = 2x$ or $y = -2x$.



The domain is the set of points (x, y) which are *not* on $y = 2x$ or $y = -2x$ – i.e. the points such that $y \neq \pm 2x$. $\square$

Example. A function of 2 variables is defined by

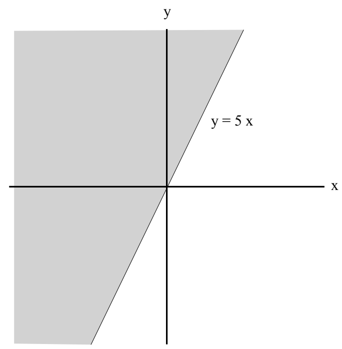
$$f(x, y) = \ln(5x - y).$$

Describe the set of points (x, y) for which f is undefined. What is the **domain** of f?

The natural log function is undefined for inputs which are less than or equal to 0. So $f(x, y) = \ln(5x - y)$ will be undefined if

$$\begin{aligned} 5x - y &\leq 0 \\ 5x &\leq y \end{aligned}$$

That is, f is undefined at points (x, y) where $y \geq 5x$. They are the points lying on or above the line $y = 5x$:



The domain is the set of points (x, y) *below* the line $y = 5x$. □

Example. A function of 2 variables is defined by

$$f(x, y) = \sqrt{1 - x^2 - y^2}.$$

Describe the set of points (x, y) for which f is undefined. What is the **domain** of f?

Since the square root of a negative number is undefined, $f(x, y) = \sqrt{1 - x^2 - y^2}$ is undefined for

$$1 - x^2 - y^2 < 0, \quad \text{or} \quad x^2 + y^2 > 1.$$

That is, f is undefined at points (x, y) lying outside the unit circle $x^2 + y^2 = 1$.
The domain is the set of points (x, y) on or inside the unit circle $x^2 + y^2 = 1$. □

Example. Describe the image (or the range) of the function $z = f(x, y) = \tan^{-1}(x + y)$.

The image of a function is the set of outputs. What are the outputs of the inverse tangent function $\tan^{-1}()$?

For any x and y,

$$-\frac{\pi}{2} < \tan^{-1}(x + y) < \frac{\pi}{2}.$$

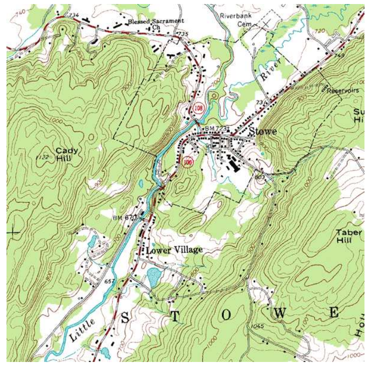
I know that $\tan^{-1} x$ attains every value between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, and if I set $y = 0$ I have $\tan^{-1}(x + y) = \tan^{-1} x$.

Therefore, the image of f is $-\frac{\pi}{2} < z < \frac{\pi}{2}$. □

A function of several variables can be pictured in many ways. For example, you can draw the **graph** of a function of 2 variables $z = f(x, y)$. Because plotting points in 3 dimensions is tedious and difficult, you'd probably use software to draw the graph.

for functions $f : \mathbb{R}^n \to \mathbb{R}$, you may also get a "picture" of the function by drawing its **level sets**. A level set for f is obtained by setting $f = c$, where c is a constant. By using different values for c, you get a picture of the "levels" of the function.

You may have seen **topographic maps**, where the level curves are referred to as **contour lines**. They represent lines along which the altitude ("z") is constant:



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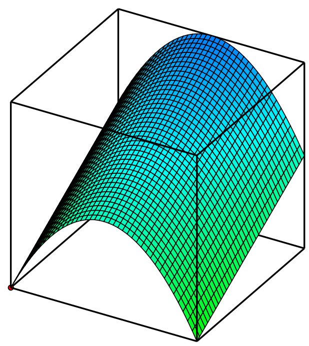
Example. Sketch the graph of $z = y - x^2$, and some of the contour lines.

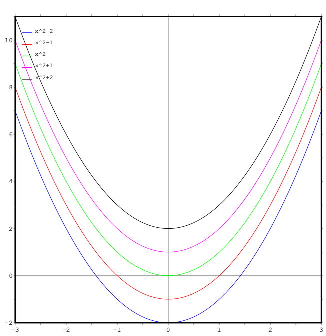
Here's the graph of the function, produced by a computer:



I can use a computer to sketch the level curves, but this example is simple enough that I'll analyze it first. I get level curves by setting z to specific numbers and graphing the curves I get.

Value of c	Equation for $z = c$
-2	$y - x^2 = -2$ or $y = x^2 - 2$
-1	$y - x^2 = -1$ or $y = x^2 - 1$
0	$y - x^2 = 0$ or $y = x^2$
1	$y - x^2 = 1$ or $y = x^2 + 1$
2	$y - x^2 = 2$ or $y = x^2 + 2$

You can see the level curves are a family of parabolas.



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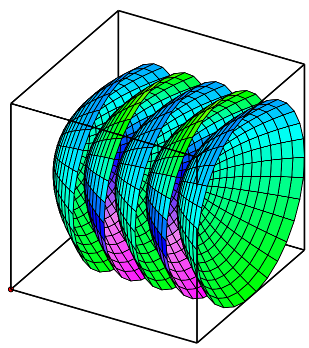
For a function of 3 variables $w = f(x, y, z)$, setting $w = c$ for various numbers c produces **level surfaces**. If you interpret $w = f(x, y, z)$ as the temperature at a point (x, y, z) in space, then a level surface $w = c$ is the set of points in space where the temperature is c.

Example. Describe some level surfaces for the function $w = x - y^2 - z^2$.

w	Equation of level surface
-4	$-4 = x - y^2 - z^2$ or $x = y^2 + z^2 + 4$
-2	$-2 = x - y^2 - z^2$ or $x = y^2 + z^2 + 2$
0	$0 = x - y^2 - z^2$ or $x = y^2 + z^2$
2	$2 = x - y^2 - z^2$ or $x = y^2 + z^2 - 2$
4	$4 = x - y^2 - z^2$ or $x = y^2 + z^2 - 4$

The level surfaces are paraboloids opening along the x-axis.



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