1-17-2022

Gradient, Divergence, and Curl

The operators named in the title are built out of the **del operator**

$$\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}.$$

(It is also called **nabla**. That always sounded goofy to me, so I will call it "del".)

Del is a formal vector; it has components, but those components have partial derivative operators ($\frac{\partial}{\partial x}$ and so on) which want to be fed functions to differentiate. So as a general rule, when you multiply FOO by del, the partial derivative operators differentiate FOO as opposed to multiplying.

First, suppose f is a function. ∇f is the **gradient** of f, sometimes denoted $\operatorname{grad} f$.

Example. Compute the gradient of $f(x, y, z) = xye^{y^2 z}$.

$$\nabla f = \hat{\imath}\frac{\partial f}{\partial x} + \hat{\jmath}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z} = \hat{\imath}\left(ye^{y^2 z}\right) + \hat{\jmath}\left(xe^{y^2 z} + 2xy^2 e^{y^2 z}\right) + \hat{k}\left(xy^3 e^{y^2 z}\right).$$

You can also write this as

$$\nabla f = \left(ye^{y^2 z}, xe^{y^2 z} + 2xy^2 e^{y^2 z}, xy^3 e^{y^2 z}\right). \quad \square$$

∇ is a vector; what kinds of things can you do with a vector? Well, you can take the dot product of the vector with another vector. If you do that with ∇, you obtain the **divergence**:

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F}.$$

For this to make sense, $\vec{F}$ should be a **vector field**; since the dot product of two vectors is a number, $\operatorname{div} \vec{F}$ should be a number (that is, a numerical function).

Example. Compute the divergence of $\vec{F} = (x^2 + y)\hat{\imath} + (y^2 - z)\hat{\jmath} + (z^2 + x)\hat{k}$.

$$\operatorname{div} \vec{F} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left((x^2 + y)\hat{\imath} + (y^2 - z)\hat{\jmath} + (z^2 + x)\hat{k}\right) =$$

$$\frac{\partial}{\partial x}(x^2 + y) + \frac{\partial}{\partial y}(y^2 - z) + \frac{\partial}{\partial z}(z^2 + x) = 2x + 2y + 2z. \quad \square$$

What is the divergence of a vector field? If you think of the field as the velocity field of a fluid flowing in three dimensions, then $\operatorname{div} \vec{F} = 0$ means the fluid is **incompressible** — for any closed region, the amount of fluid flowing in through the boundary equals the amount flowing out. This result follows from the **Divergence Theorem**, one of the big theorems of vector integral calculus.

You can take the **cross product** of two 3-dimensional vectors; if you do this with ∇, you obtain the **curl** of a vector field:

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F}.$$

More specfically, suppose $\vec{F} = (F_1, F_2, F_3)$. Then

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.$$

The cross product of two vectors is a vector, so curl takes a vector field to another vector field.

Example. Compute the curl of $\vec{F} = (x^2 + y)\hat{\imath} + (y^2 - z)\hat{\jmath} + (z^2 + x)\hat{k}$.

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 + y & y^2 - z & z^2 + x \end{vmatrix}.$$

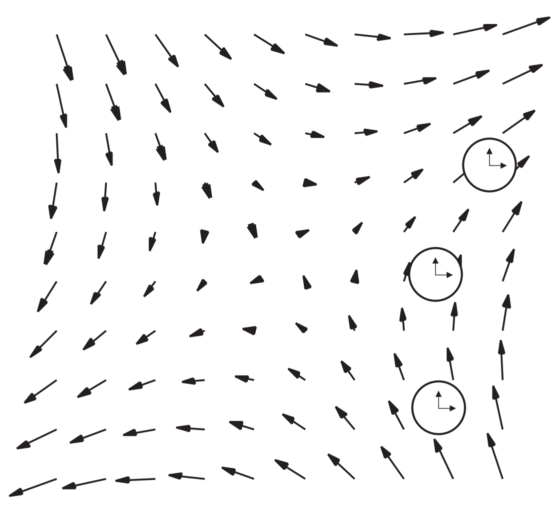
As usual, I'll expand the determinant by cofactors of the first row:

$$\hat{\imath} \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 - z & z^2 + x \end{vmatrix} - \hat{\jmath} \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ x^2 + y & z^2 + x \end{vmatrix} + \hat{k} \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ x^2 + y & y^2 - z \end{vmatrix} =$$

$$\hat{\imath}\left(\frac{\partial}{\partial y}(z^2 + x) - \frac{\partial}{\partial z}(y^2 - z)\right) - \hat{\jmath}\left(\frac{\partial}{\partial x}(z^2 + x) - \frac{\partial}{\partial z}(x^2 + y)\right) + \hat{k}\left(\frac{\partial}{\partial x}(y^2 - z) - \frac{\partial}{\partial y}(x^2 + y)\right) =$$

$$(0 - (-1))\hat{\imath} - (1 - 0)\hat{\jmath} + (0 - 1)\hat{k} = \hat{\imath} - \hat{\jmath} + \hat{k} = (1, -1, 1). \quad \square$$

What is the curl of a vector field? To make it easier to visualize, suppose $\vec{F}$ is the velocity field for a fluid flow *in the plane* (so the z component is 0).



Drop a marked float into the flow and let it be carried along by the fluid. Then $\operatorname{curl} \vec{F} = \vec{0}$ means, roughly, that the float doesn't rotate as it moves.

The old name for $\operatorname{curl} \vec{F}$ was $\operatorname{rot} \vec{F}$, the **rotation** of $\vec{F}$. A field with $\operatorname{rot} \vec{F} = \vec{0}$ is said to be **irrotational** (and this terminology is still used, even though rot has been replaced by curl).

There are many identities involving div, grad, and curl; here is an important one.

Example. Show that $\operatorname{curl}\operatorname{grad} f = \vec{0}$.

$$\operatorname{curl}\operatorname{grad} f = \nabla \times \nabla f = \left|\begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{matrix}\right|.$$

You can grind this out and discover that everything cancels by equality of mixed partial derivatives (e.g. $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$). But it's easier to operate formally, factoring f out of the last row:

$$\left|\begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{matrix}\right| = \left|\begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \end{matrix}\right| (f).$$

But the determinant of a matrix with two equal rows is 0, so the result is $\vec{0}$. $\square$