4-1-2018

Interchanging the Order of Integration

Consider the iterated integral

$$\iint_D f(x,y)\,dx\,dy.$$

It can be computed by integrating with respect to x first or with respect to y first. In some cases, one order is better than the other. For this reason, it's useful to know how to go from a "bad" order of integration to a "good" order of integration.

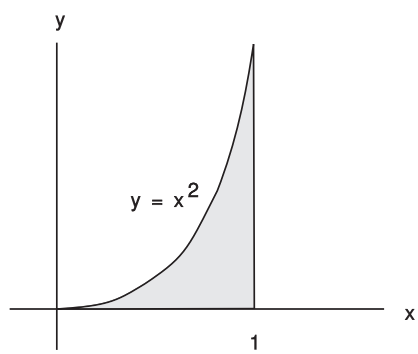
Example. Compute $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 \frac{1}{\sqrt{x^3+1}}\,dx\,dy$.

As the integral is given, I'd need to integrate first with respect to x. However, I don't know the antiderivative of $\dfrac{1}{\sqrt{x^3+1}}$. I'll interchange the order of integration and integrate first with respect to y.

Step 1: Pull off the limits of integration as inequalities.

$$\left\{\begin{matrix} 0 \le y \le 1 \\ \sqrt{y} \le x \le 1 \end{matrix}\right\}$$

Step 2: Draw the region defined by the inequalities.



Step 3: Describe the region by inequalities with the variables in the opposite order.

In the first set of inequalities, y came first. In this set, x will come first. For x, I can take the numerical bounds in the x-direction: $0 \le x \le 1$.

Next, I need the inequalities for y. y is the *vertical* variable, so it will be bounded by expressions for the *bottom curve* and the *top curve* of the region. The bottom curve is the x-axis, which is $y = 0$. The top curve is $x = \sqrt{y}$. Since I'm bounding y, I need to express y in terms of x. Thus, $y = x^2$.

Therefore, the inequalities for y are $0 \le y \le x^2$. The new set of inequalities is

$$\left\{\begin{matrix} 0 \le x \le 1 \\ 0 \le y \le x^2 \end{matrix}\right\}$$

Step 4: Put the inequalities back onto the integral:

$$\int_0^1 \int_0^{x^2} \frac{1}{\sqrt{x^3+1}}\,dy\,dx = \int_0^1 \frac{1}{\sqrt{x^3+1}} \int_0^{x^2} dy\,dx = \int_0^1 \frac{1}{\sqrt{x^3+1}}\,[y]_0^{x^2}\,dx =$$

$$\int_0^1 \frac{x^2}{\sqrt{x^3+1}}\,dx = \left[\frac{2}{3}\sqrt{x^3+1}\right]_0^1 = \frac{2}{3}(\sqrt{2}-1) = 0.27614\ldots. \quad \square$$

Schematically, here's what's going on:

integral → inequalities → picture → inequalities → integral

This is similar to the procedure for converting a double integral to polar coordinates.

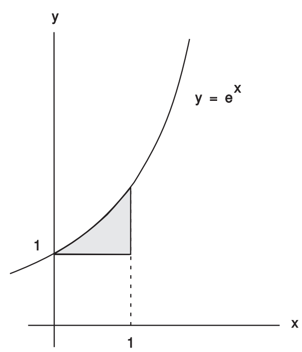
Example. Compute the integral by interchanging the order of integration:

$$\int_1^e \int_{\ln y}^1 \cos(e^x - x)\,dx\,dy.$$

Pull off the limits as inequalities:

$$\left\{\begin{array}{c} 1 \le y \le e \\ \ln y \le x \le 1 \end{array}\right\}$$

Next, draw the region determined by the inequalities. The inequalities $1 \le y \le e$ imply that the region lies in the horizontal strip between $y = 1$ (bottom) and $y = e$ (top).



The inequalities $\ln y \le x \le 1$ give the *left-hand* and *right-hand* boundaries, because x is the horizontal variable. The left-hand curve is $x = \ln y$, or $y = e^x$. The right-hand curve is $x = 1$. The region is shown above.

Next, describe the region by inequalities with the variables switched. I'll do x first, since the first set of inequalities had the number bounds on y. The numerical bounds on x are 0 and 1, so $0 \le x \le 1$.

To get the bounds on y, I look at the *bottom curve* and the *top curve*. The bottom curve is the line $y = 1$. The top curve is $y = e^x$. Hence, the inequalities for y are $1 \le y \le e^x$.

The new inequalities are

$$\left\{\begin{array}{c} 0 \le x \le 1 \\ 1 \le y \le e^x \end{array}\right\}$$

Put the inequalities back onto the integral:

$$\int_0^1 \int_1^{e^x} \cos(e^x - x)\,dy\,dx = \int_0^1 \cos(e^x - x) \int_1^{e^x} dy\,dx = \int_0^1 \cos(e^x - x)\,[y]_1^{e^x}\,dx =$$

$$\int_0^1 (e^x - 1)\cos(e^x - x)\,dx = \int_1^{e-1} (e^x - 1)\cos u \cdot \frac{du}{e^x - 1} = \int_1^{e-1} \cos u\,du = [\sin u]_1^{e-1} =$$

$$\left[u = e^x - x, \quad du = (e^x - 1)\,dx, \quad dx = \frac{du}{e^x - 1}; x = 0, \quad u = 1; \quad x = 1, \quad u = e - 1\right]$$

$$\sin(e-1) - \sin 1 = 0.14767\ldots. \quad \square$$

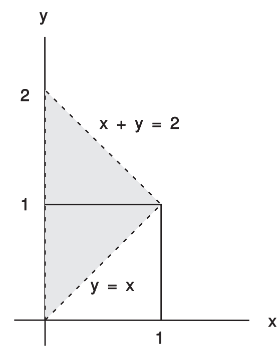
Example. Express the following sum as a single iterated integral by interchanging the order of integration:

$$\int_0^1 \int_0^y f(x,y)\,dx\,dy + \int_1^2 \int_0^{2-y} f(x,y)\,dx\,dy$$

Pull off the limits as inequalities:

$$\left\{\begin{matrix} 0 \le y \le 1 \\ 0 \le x \le y \end{matrix}\right\} \quad \text{and} \quad \left\{\begin{matrix} 1 \le y \le 2 \\ 0 \le x \le 2 - y \end{matrix}\right\}$$

Draw the region determined by the inequalities.



Describe the region by inequalities with the variables switched:

$$\left\{\begin{matrix} 0 \le x \le 1 \\ x \le y \le 2 - x \end{matrix}\right\}$$

Put the new inequalities back onto the integral:

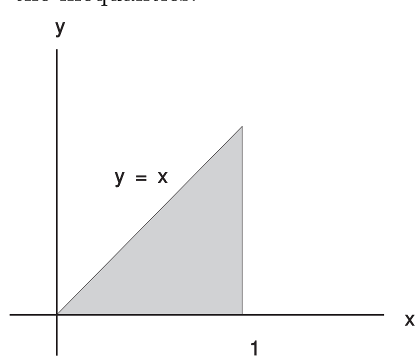
$$\int_0^1 \int_x^{2-x} f(x,y)\,dy\,dx. \quad \square$$

Example. Compute $\displaystyle\int_0^1 \int_y^1 2\cos(x^2)\,dx\,dy$.

Pull off the limits as inequalities:

$$\left\{\begin{matrix} 0 \le y \le 1 \\ y \le x \le 1 \end{matrix}\right\}$$

Draw the region determined by the inequalities.



Describe the region by inequalities with the variables switched:

$$\left\{\begin{array}{c} 0 \le x \le 1 \\ 0 \le y \le x \end{array}\right\}$$

Put the new inequalities back onto the integral:

$$\int_0^1 \int_y^1 2\cos(x^2)\,dx\,dy = \int_0^1 \int_0^x 2\cos(x^2)\,dydx = \int_0^1 2x\cos(x^2)\,dx =$$

$$\left[u = x^2, \quad du = 2x\,dx, \quad dx = \frac{du}{2x}; \quad x = 0, \quad u = 0; \quad x = 1, \quad u = 1\right]$$

$$\int_0^1 2x\cos u \cdot \frac{du}{2x} = \int_0^1 \cos u\,du = [\sin u]_0^1 = \sin 1 = 0.84147\ldots. \quad \square$$
