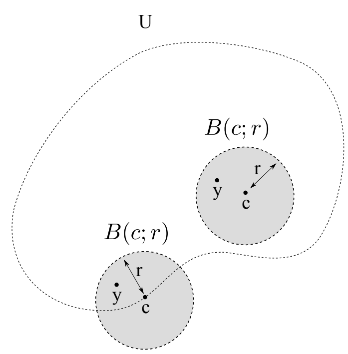
Limits and Continuity

I'll give the precise definition of a limit so that you can see the similarity to the definition you saw in single-variable calculus. The first definition is a technical point which you don't need to worry about too much. It simply ensures if we take a limit as $x \to c$, that x can approach c through a set where the function is defined.

Definition. Let U be a subset of $\mathbb{R}^n$. A point $c \in \mathbb{R}^n$ is an **accumulation point** of U if for every $r > 0$, the open ball $B(c; r)$ contains a point $y \in U$ other than c.

$B(c; r)$ is the set of points in $\mathbb{R}^n$ which are less than r units from c:

$$B(c; r) = \{x \in \mathbb{R}^n \mid r > \|x - c\|\}.$$



Definition. Let $f : U \to \mathbb{R}^m$ be a function defined on $U \subset \mathbb{R}^n$, and let c be an accumulation point of U. Then $\lim\limits_{x \to c} f(x) = L$ means:

For every $\epsilon > 0$, there is a δ, such that

$$\text{if} \quad \delta > \|x - c\| > 0, \quad \text{then} \quad \epsilon > \|f(x) - L\|.$$

Many results you know about limits from single-variable calculus have analogs for functions of several variables.

Proposition. Suppose $f, g : U \to \mathbb{R}$ where $U \subset \mathbb{R}^n$. Let c be an accumulation point of U, and let $k \in \mathbb{R}$. Then:

(a) $\lim\limits_{x \to c} [f(x) + g(x)] = \lim\limits_{x \to c} f(x) + \lim\limits_{x \to c} g(x)$.

(b) $\lim\limits_{x \to c} k \cdot f(x) = k \cdot \lim\limits_{x \to c} f(x)$.

(c) $\lim\limits_{x \to c} f(x) \cdot g(x) = [\lim\limits_{x \to c} f(x)][\lim\limits_{x \to c} g(x)]$.

(d) $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$, provided that $\lim\limits_{x \to c} g(x) \neq 0$.

All of these results mean that if the limits on the right side are defined, then so is the limit on the left side, and the two sides are equal. □

I won't try to state all of the easy results on limits that generalize to functions of several variables. You will see many of them proven in a course in real analysis. Let's look at some of the complications that result from being able to approach a point in more than one dimension.

Example. Compute $\lim\limits_{(x,y) \to (-1,2)} \dfrac{5x + y + 1}{2x - y}$.

I can compute the limit by plugging in. (This is another way of saying that $f(x,y) = \dfrac{5x+y+1}{2x-y}$ is continuous at $(3,1)$.) Thus.

$$\lim_{(x,y)\to(3,1)} \frac{5x+y+1}{2x-y} = \frac{5\cdot 3+1+1}{2\cdot 3-1} = \frac{17}{5}. \quad \square$$

Example. Compute $\displaystyle\lim_{(x,y,z)\to(4,1,2)} (x+3y)e^{-z}$.

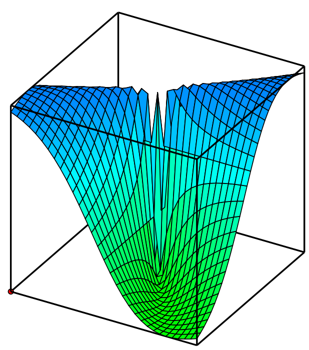
I can compute the limit by plugging in.

$$\lim_{(x,y,z)\to(4,1,2)} (x+3y)e^{-z} = (4+3\cdot 1)e^{-2} = 7e^{-2}. \quad \square$$

Example. Compute $\displaystyle\lim_{(x,y)\to(0,0)} \frac{5xy-2y^2}{x^2+y^2}$.

Substituting $(x,y)=(0,0)$ yields the indeterminate form $\dfrac{0}{0}$.

Here's the graph of the function. Notice that as $(x,y)\to(0,0)$ the height seems to approach one value along the "ridgeline" while it approaches another value along the "valley":



This leads me to believe that the limit is undefined.

To prove this, I try to find different ways of approaching $(0,0)$ which give different limits. Specifically, I try to pick different curves through $(0,0)$ which make $\dfrac{5xy-2y^2}{x^2+y^2}$ simplify to different values. In this case, I try the x-axis, which is $y=0$, and the y-axis, which is $x=0$.

Set $y=0$ and let $x\to 0$. I have

$$\lim_{x\to 0} \frac{5x\cdot 0-2\cdot 0^2}{x^2+0^2} = \lim_{x\to 0} \frac{0}{x^2} = \lim_{x\to 0} 0 = 0.$$

Set $x=0$ and let $y\to 0$. I have

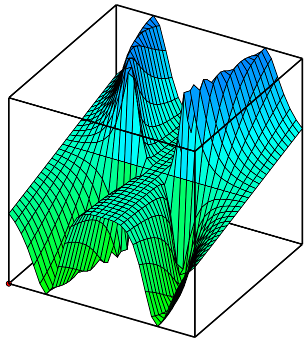
$$\lim_{y\to 0} \frac{5\cdot 0\cdot y-2y^2}{0^2+y^2} = \lim_{y\to 0} \frac{-2y^2}{y^2} = \lim_{y\to 0}(-2) = -2.$$

Since $\dfrac{5xy-2y^2}{x^2+y^2}$ approaches different numbers depending on how (x,y) approaches $(0,0)$, the limit is undefined. $\square$

Example. Compute $\displaystyle \lim_{(x,y)\to(0,0)} \frac{3x^4y}{x^8+y^2}$.

Substituting $(x,y)=(0,0)$ yields the indeterminate form $\dfrac{0}{0}$.

Here's the graph of the function. It is a little harder to tell from the graph what is happening near the origin.



It turns out that the limit is undefined. To show this, I'll approach $(0,0)$ along a line and along a curve.

If you approach $(0,0)$ along the line $y=0$, you get

$$\lim_{(x,y)\to(0,0)} \frac{3x^4y}{x^8+y^2} = \lim_{x\to 0} \frac{3\cdot x^4\cdot 0}{x^8+0^2} = \lim_{x\to 0} \frac{0}{x^8} = \lim_{x\to 0} 0 = 0$$

Next, I notice that $x^4\cdot x^4 = x^8$ and $(x^4)^2 = x^8$. Thus, I can get multiple "x^8" terms by setting $y=x^4$.

If you approach $(0,0)$ along the curve $y=x^4$, you get

$$\lim_{(x,y)\to(0,0)} \frac{3x^4y}{x^8+y^2} = \lim_{x\to 0} \frac{3x^4\cdot x^4}{x^8+(x^4)^2} = \lim_{x\to 0} \frac{3x^8}{2x^8} = \lim_{x\to 0} \frac{3}{2} = \frac{3}{2}.$$

Since $\dfrac{3x^4y}{x^8+y^2}$ approaches different numbers depending on how (x,y) approaches $(0,0)$, the limit is undefined. □

Example. Compute $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$ by converting to polar coordinates.

Let $x = r\cos\theta$ and $y = r\sin\theta$. Then

$$\frac{x^2y^3}{x^2+y^2} = \frac{(r\cos\theta)^2(r\sin\theta)^2}{(r\cos\theta)^2+(r\sin\theta)^2} = \frac{r^4(\cos\theta)^2(\sin\theta)^2}{r^2} = r^2(\cos\theta)^2(\sin\theta)^2.$$

Then

$$\lim_{(x,y)\to(0,0)} \frac{2xy^3}{x^2+y^2} = \lim_{r\to 0} r^2(\cos\theta)^2(\sin\theta)^2 = 0. \quad \square$$

Example. Compute $\displaystyle \lim_{(x,y,z)\to(0,0,0)} \frac{x^2+y^2+z^2}{5y^2+z^2}$.

I try to find different curves through $(0,0)$ which make $\dfrac{x^2+y^2+z^2}{5y^2+z^2}$ simplify to different values.

If you approach $(0,0,0)$ along the line $y = z$, $x = 0$, you get

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^2+y^2+z^2}{5y^2+z^2} = \lim_{z\to 0} \frac{0^2+z^2+z^2}{5z^2+z^2} = \lim_{z\to 0} \frac{2z^2}{6z^2} = \lim_{z\to 0} \frac{1}{3} = \frac{1}{3}.$$

If you approach $(0,0,0)$ along the line $x = y = z$, you get

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^2+y^2+z^2}{5y^2+z^2} = \lim_{x\to 0} \frac{x^2+x^2+x^2}{5x^2+x^2} = \lim_{x\to 0} \frac{3x^2}{6x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}.$$

Since you get different limits by approaching $(0,0,0)$ in different ways, $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^2+y^2+z^2}{5y^2+z^2}$ is undefined. $\square$

Definition. Let $f : U \to \mathbb{R}^m$, where $U \subset \mathbb{R}^n$, and let $c \in U$. Then f is **continuous** at c if

$$\lim_{x\to c} f(x) = f(c).$$

Remark. Some authors will say a function $\mathbb{R}^n \to \mathbb{R}^m$ is not continous at a point where the function isn't defined. For example, the function f defined by $f(x) = \dfrac{1}{x-1}$ has (natural) domain $x \ne 1$. These authors will say that f is not continuous at $x = 1$, with similar terminology for multivariable functions. I'll avoid doing this: It seems inappropriate to talk about whether a function does or does not have a property like continuity at the point where there is no function!

I'll only consider continuity (or lack of continuity) at points in a function's domain.

Example. A function $\mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x,y) = \begin{cases} \dfrac{x+3y}{2x+5y} & \text{if } (x,y) \ne (2,1) \\ \dfrac{4}{9} & \text{if } (x,y) = (2,1) \end{cases}.$$

Is f continuous at $(2,1)$?

$$\lim_{(x,y)\to(2,1)} f(x,y) = \lim_{(x,y)\to(2,1)} \frac{x+3y}{2x+5y} = \frac{5}{9}.$$

However, $f(2,1) = \dfrac{4}{9}$,

Since $\displaystyle\lim_{(x,y)\to(2,1)} f(x,y) \ne f(2,1)$, it follows that f is not continuous at $(2,1)$. $\square$
