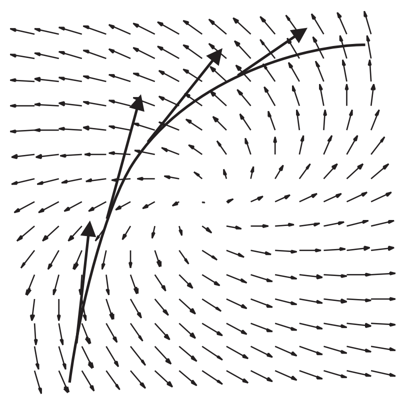
Line Integrals

If $\vec{F}$ is a vector field and $\vec{\sigma}(t)$, $a \le t \le b$, is a path, the **line integral** of $\vec{F}$ along $\vec{\sigma}(t)$ is

$$\int_{\vec{\sigma}} \vec{F} \cdot \overrightarrow{ds} = \int_a^b \vec{F}(t) \cdot \vec{\sigma}\,'(t)\,dt.$$

Here is what this looks like in two dimensions.



The vector field gives a vector at each point of the plane. As you move along the path, at each point you compute the dot product of the velocity vector $\vec{\sigma}'(t)$ with the vector from the field at that point. Integrate to "add up" the results; the total is the line integral.

Notice that I'm writing $\overrightarrow{ds}$ instead of ds, the differential for a path integral. Here's the difference:

$$\overrightarrow{ds} = \vec{\sigma}\,'(t)\,dt, \quad \text{while} \quad ds = \|\vec{\sigma}\,'(t)\|\,dt.$$

One is a vector, the other is a scalar: $\overrightarrow{ds}$ uses the velocity vector, while ds uses the *length* of the velocity vector.

Example. Compute $\displaystyle\int_{\vec{\sigma}} \vec{F} \cdot \overrightarrow{ds}$ for $\vec{F}(x, y) = (y, x)$:

(a) Along the path $\vec{\sigma}(t) = (\cos t, \sin t)$ for $0 \le t \le \dfrac{\pi}{2}$.

(b) Along the path $\vec{\tau}(t) = (\cos 2t, \sin 2t)$ for $0 \le t \le \dfrac{\pi}{4}$.

(a) The path is the part of $x^2 + y^2 = 1$ lying in the first quadrant — a quarter circle. $\vec{\sigma}(0) = (1, 0)$ while $\vec{\sigma}\left(\dfrac{\pi}{2}\right) = (0, 1)$, so the quarter circle is traversed counterclockwise.

The velocity is

$$\vec{\sigma}\,'(t) = (-\sin t, \cos t).$$

Write the vector field in terms of t:

$$\vec{F}(t) = (y, x) = (\sin t, \cos t).$$

Then

$$\vec{F}(t) \cdot \vec{\sigma}\,'(t) = -(\sin t)^2 + (\cos t)^2 = \cos 2t.$$

(I used a double angle formula from trigonometry.) Therefore,

$$\int_{\vec{\sigma}} \vec{F} \cdot \overrightarrow{ds} = \int_0^{\pi/2} \cos 2t\,dt = \left[\frac{1}{2}\sin 2t\right]_0^{\pi/2} = 0. \quad \square$$

(b) τ is the same quarter circle as in (a), but

$$\vec{\tau}\,'(t) = (-2\sin 2t, 2\cos 2t).$$

The extra factor of 2 means the path is traversed twice as fast as (a). The field is

$$\vec{F}(t) = (y, x) = (\sin 2t, \cos 2t).$$

So

$$\vec{F}(t) \cdot \vec{\tau}\,'(t) = -2(\sin 2t)^2 + 2(\cos 2t)^2 = 2\cos 4t.$$

Therefore,

$$\int_{\vec{\tau}} \vec{F} \cdot \overrightarrow{ds} = \int_0^{\pi/4} 2\cos 4t\, dt = \left[\frac{1}{2}\sin 4t\right]_0^{\pi/4} = 0.$$

Traversing the path twice as rapidly made no difference. This is true in general: *If you traverse the same path in the same direction at different speeds, the line integral does not change.* □

Example. Compute $\displaystyle\int_{\vec{\sigma}} \vec{F} \cdot \overrightarrow{ds}$ for $\vec{F}(x, y) = (x + y, x - y)$:

(a) Along the path $\vec{\sigma}(t) = (t, t^2)$ for $0 \le t \le 1$.

(b) Along the path $\vec{\tau}(t) = (1 - t, (1 - t)^2)$ for $0 \le t \le 1$.

(a) The path is the part of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$. I have

$$\vec{\sigma}\,'(t) = (1, 2t) \quad \text{and} \quad \vec{F}(x, y) = (x + y, x - y) = (t + t^2, t - t^2).$$

Hence,

$$\vec{F}(t) \cdot \vec{\sigma}\,'(t) = t + 3t^2 - 2t^3.$$

The integral is

$$\int_{\vec{\sigma}} \vec{F} \cdot \overrightarrow{ds} = \int_0^1 (t + 3t^2 - 2t^3)\, dt = \left[\frac{1}{2}t^2 + t^3 - \frac{1}{2}t^4\right]_0^1 = 1. \quad □$$

(b) If I change the path to $\vec{\tau}(t) = (1 - t, (1 - t)^2)$, I am now traversing $y = x^2$ from $(1, 1)$ to $(0, 0)$ — the same path as in (a), but in the opposite direction.

I have

$$\vec{\tau}\,'(t) = (-1, -2 + 2t).$$

$$\vec{F}(t) = ((1 - t) + (1 - t)^2, (1 - t) - (1 - t)^2) = (t^2 - 3t + 2, t - t^2).$$

Hence,

$$\vec{F}(t) \cdot \vec{\tau}\,'(t) = -2t^3 + 3t^2 + t - 2.$$

Therefore,

$$\int_{\vec{\tau}} \vec{F} \cdot \overrightarrow{ds} = \int_0^1 (-2t^3 + 3t^2 + t - 2)\, dt = \left[-\frac{1}{2}t^4 + t^3 + \frac{1}{2}t^2 - 2t\right]_0^1 = -1.$$

This is true in general: *If you traverse the same path in the opposite direction, the line integral is multiplied by* -1. □

Differential notation.

If $\vec{F} = (F_1, F_2, F_3)$ is a vector field and $\vec{\sigma}(t)$ for $a \le t \le b$, is a path, then

$$\vec{F} \cdot \vec{\sigma}'(t) = (F_1, F_2, F_3) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}.$$

So

$$\int_{\vec{\sigma}} \vec{F} \cdot \overrightarrow{ds} = \int_a^b \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}\right) dt.$$

If I formally multiply the dt in, I get

$$\int_{\vec{\sigma}} \vec{F} \cdot \overrightarrow{ds} = \int_a^b (F_1\, dx + F_2\, dy + F_3\, dz).$$

You will sometimes see line integrals written in this form.

You can do the computation converting F_1, F_2, and F_3 to functions of t using $\vec{\sigma} = (x(t), y(t), z(t))$. Replace dx, dy, dz by $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, and integrate the whole thing with respect to t.

Alternatively, you can use one of the coordinate variables x, y, or z as the parameter.

Example. Let $\vec{\sigma}(t)$ be the segment joining $(1,1,1)$ to $(2,3,4)$. Compute

$$\int_{\vec{\sigma}} x\, dx + y\, dy + (x + y - 1)\, dz.$$

The segment is

$$\vec{\sigma}(t) = (1-t) \cdot (1,1,1) + t \cdot (2,3,4) = (1+t, 1+2t, 1+3t).$$

That is,

$$x = 1 + t, \quad y = 1 + 2t, \quad z = 1 + 3t.$$

The parameter range is $0 \le t \le 1$.

Now

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 2, \quad \frac{dz}{dt} = 3.$$

Plugging everything into the integral, I get

$$\int_{\vec{\sigma}} x\, dx + y\, dy + (x + y - 1)\, dz = \int_0^1 [(1+t)(1) + (1+2t)(2) + (1+3t)(3)]\ dt = \int_0^1 (6 + 14t)\, dt =$$

$$\left[6t + 7t^2\right]_0^1 = 13. \quad \square$$

Example. Compute $\int_C (x + 8y)\, dx + 5x^2\, dy$ where C is the part of the curve $y = x^3$ going from $(0,0)$ to $(1,1)$.

I'll do everything in terms of x. I have $\frac{dy}{dx} = 3x^2$, so dy becomes $3x^2\, dx$. The curve extends from $x = 0$ to $x = 1$. Substituting $y = x^3$ and $dy = 3x^2\, dx$, I get

$$\int_C (x + 8y)\, dx + 5x^2\, dy = \int_0^1 [(x + 8x^3) + 5x^2 \cdot 3x^2]\, dx = \int_0^1 (x + 8x^3 + 15x^4)\, dx =$$

$$\left[\frac{1}{2}x^2 + 2x^4 + 3x^5\right]_0^1 = \frac{11}{2}. \quad \square$$

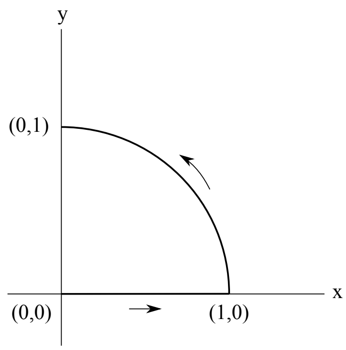
If $\vec{F}$ represents a **force field** and $\vec{\sigma}$ is the tranjectory of an object moving in the field, the **work** done by the object in moving along the path through the field is

$$\int_{\vec{\sigma}} \vec{F} \cdot \overrightarrow{ds}.$$

Example. Suppose a force field is given by

$$\vec{F}(x, y) = (x + y, 2y).$$

Find the work done by a particle moving along a path consisting of the segment from $(0, 0)$ to $(1, 0)$, followed by the arc of the circle $x^2 + y^2 = 1$ from $(1, 0)$ counterclockwise to $(0, 1)$.



The segment from $(0, 0)$ to $(1, 0)$ may be parametrized by

$$\vec{\sigma}(t) = (t, 0) \quad \text{for} \quad 0 \le t \le 1.$$

Thus,

$$x = t, \quad y = 0.$$

Hence,

$$\vec{\sigma}\,'(t) = (1, 0).$$

On this segment,

$$\vec{F}(t) = (t, 0).$$

So

$$\vec{F} \cdot \vec{\sigma}\,'(t) = (t, 0) \cdot (1, 0) = t.$$

The work done is

$$\int_{\vec{\sigma}} \vec{F} \cdot \overrightarrow{ds} = \int_0^1 t\,dt = \left[\frac{1}{2}t^2\right]_0^1 = \frac{1}{2}.$$

The arc of the circle from $(1, 0)$ to $(0, 1)$ may be parametrized by

$$\vec{\tau}(t) = (\cos t, \sin t) \quad \text{for} \quad 0 \le t \le \frac{\pi}{2}.$$

Thus,

$$x = \cos t, \quad y = \sin t.$$

Hence,

$$\vec{\tau}\,'(t) = (-\sin t, \cos t).$$

On this arc,

$$\vec{F}(t) = (\cos t + \sin t, 2\sin t).$$

So

$$\vec{F} \cdot \vec{\tau}\,'(t) = (\cos t + \sin t, 2\sin t) \cdot (-\sin t, \cos t) = -\cos t \sin t - (\sin t)^2 + 2\sin t \cos t = \sin t \cos t - (\sin t)^2.$$

The work done is

$$\int_{\vec{\tau}} \vec{F} \cdot \overrightarrow{ds} = \int_0^{\pi/2} (\sin t \cos t - (\sin t)^2)\, dt = \int_0^{\pi/2} \left(\frac{1}{2}\sin 2t - \frac{1}{2}(1 - \cos 2t)\right)\, dt =$$

$$\left[-\frac{1}{4}\cos 2t - \frac{1}{2}t + \frac{1}{4}\sin 2t\right]_0^{\pi/2} = \frac{1}{2} - \frac{\pi}{4} = -0.28539\ldots. \quad \square$$
