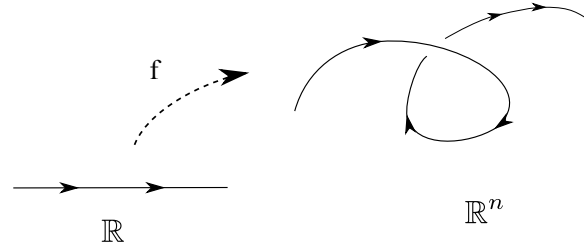# Parametric Curves

A vector function $f : \mathbb{R} \to \mathbb{R}^n$ can be thought of as a curve in $\mathbb{R}^n$. If you think of the real line as a wire, f takes the wire and bends and distorts it, then places it in $\mathbb{R}^n$.

I'll be a little sloppy and refer to both the function f and the set of output points in $\mathbb{R}^n$ (what mathematicians call the **image** of f, denoted im f) as "the curve".

To make things easier to describe, suppose $f : \mathbb{R} \to \mathbb{R}^3$, so f is a curve in space. Write f in component form:

$$f(t) = (f_1(t), f_2(t), f_3(t)).$$

You can describe f in terms of its component functions, which give the x, y, and z coordinates of a point on the curve:

$$x = f_1(t), \quad y = f_2(t), \quad z = f_3(t).$$

These are the **parametric equations** for the curve. I'll look at examples of how you can obtain parametric equations for various curves.

Example. Find parametric equations for the segment from $P(3, 9, -5)$ to $Q(1, 17, -3)$.

The **segment** from a point P to a point Q is given by

$$(x, y, z) = (1 - t) \cdot P + t \cdot Q.$$

I'm giving the formula for points in $\mathbb{R}^3$, but the same formula works for points in $\mathbb{R}^n$ for any n.

You get the segment from P to Q if you use the range $0 \le t \le 1$. If you use the range $\infty < t < \infty$, you get the line through P and Q. This gives another way to obtain the parametric equations for a line.

In this example, I have

$$\begin{aligned}
(x, y, z) &= (1 - t) \cdot (3, 9, -5) + t \cdot Q(1, 17, -3) \\
&= (3(1 - t), 9(1 - t), -5(1 - t)) + (t, 17t, -3t) \\
&= (3 - 2t, 9 + 8t, -5 + 2t)
\end{aligned}$$

The parametric equations are

$$x = 3 - 2t, \quad y = 9 + 8t, \quad z = -5 + 2t. \quad \square$$

In many cases, we'll need equations of circles or ellipses. You can obtain parametrizations for these curves using the identity

$$(\cos t)^2 + (\sin t)^2 = 1.$$

Example. Find parametric equations for the circle

$$(x+7)^2 + (y-4)^2 = 36.$$

Divide both sides by 36 and write the terms on the left as squares:

$$\frac{(x+7)^2}{36} + \frac{(y-4)^2}{36} = 1$$

$$\left(\frac{x+7}{6}\right)^2 + \left(\frac{y-4}{6}\right)^2 = 1$$

Match this against the identity

$$(\cos t)^2 + (\sin t)^2 = 1.$$

I have

$$\frac{x+7}{6} = \cos t \quad \text{and} \quad \frac{y-4}{6} = \sin t.$$

Solving these equations for x and y give the parametric equations:

$$x = -7 + 6\cos t, \quad y = 4 + 6\sin t. \quad \square$$

Example. Find parametric equations for the ellipse

$$4(x-1)^2 + 16(y-5)^2 = 49.$$

Divide both sides by 49 and write the terms on the left as squares:

$$\frac{4(x-1)^2}{49} + \frac{16(y-5)^2}{49} = 1$$

$$\left(\frac{2(x-1)}{7}\right)^2 + \left(\frac{4(y-5)}{7}\right)^2 = 1$$

Match this against the identity

$$(\cos t)^2 + (\sin t)^2 = 1.$$

I have
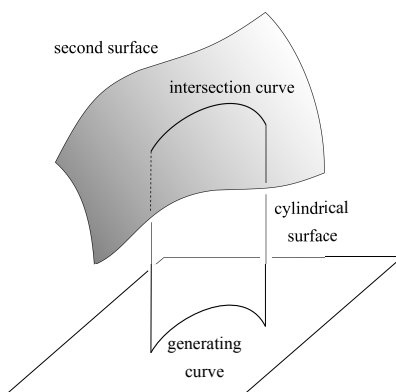
$$\frac{2(x-1)}{7} = \cos t \quad \text{and} \quad \frac{4(y-5)}{7} = \sin t.$$

Solving these equations for x and y give the parametric equations:

$$x = 1 + \frac{7}{2}\cos t, \quad y = 5 + \frac{7}{4}\sin t. \quad \square$$

Often a curve appears as the intersection of two surfaces. Suppose one of the surfaces is simply the "cylinder" generated by a curve in one of the coordinate planes (that is, you get the surface by moving the

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curve perpendicular to the plane).

You can often obtain a parametrization in the following way:

1. Parametrize the curve which generates the cylindrical surface: This gives you equations for the coordinate variables for the plane containing the curve.

2. Plug the curve parametrization into the equation for the second surface and solve for the remaining variable.

Example. Parametrize the curve of intersection of the elliptical cylinder $4x^2 + y^2 = 9$ with the plane $3x - 8y + z = 9$.

Thinking of $4x^2 + y^2 = 9$ as an ellipse in the x-y plane, I may parametrize the curve by

$$x = \frac{3}{2}\cos t, \quad y = 3\sin t.$$

For the plane, $z = -3x + 8y + 9$, so plugging in the expressions above for x and y gives

$$z = -\frac{9}{2}\cos t + 24\sin t + 9.$$

The curve is
$$x = \frac{3}{2}\cos t, \quad y = 3\sin t, \quad z = -\frac{9}{2}\cos t + 24\sin t + 9. \quad \square$$

Sometimes you can simply solve the equations for the intersecting surfaces simultaneously, giving a curve whose equation you can parametrize.

Example. Parametrize the curve of intersection of the plane $x = 1$ with the surface $z = x^2 - 2x^2y + y^2$.

The intersection of the surfaces is obtained by setting $x = 1$ in $z = x^2 - 2x^2y + y^2$:

$$z = 1^2 - 2 \cdot 1^2 \cdot y + y^2 = 1 - 2y + y^2.$$
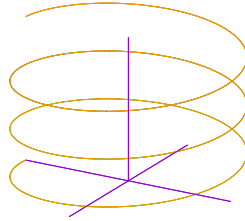
Set $y = t$, so $z = 1 - 2t + t^2$.
The parametrization is
$$x = 1, \quad y = t, \quad z = 1 - 2t + t^2. \quad \square$$

In some cases, parametrizations arise independently of surfaces.

Example. Sketch the graph of

$$x = \cos t, \quad y = \sin t, z = t.$$

The graph is a **helix**:

We've seen that the $x = \cos t$, $y = \sin t$ pair gives a circular movement or a rotation. As t increases, z (the height above the x-y-plane) increases. So the curve goes around the z-axis, rising as it goes. $\square$