4-2-2024

Parametric Surfaces

I described a **surface** as a 2-dimensional object in space. Here is a more precise definition.

Definition. A **surface** in $\mathbb{R}^3$ is a function $S : \mathbb{R}^2 \to \mathbb{R}^3$. If u and v are the input variables (often called **parameters**) and x, y, and z are the output variables, then S can be written in component form as

$$x = f(u, v), \quad y = g(u, v), \quad z = h(u, v).$$

This is called a **parametrization** of the surface, or you might describe S as a **parametric surface**.

In most cases, we'll assume that f, g, and h are continuous, or perhaps even differentiable; we'll discuss continuity and differentiability for functions of several variables later.

Example. Consider the surface given by

$$S(u, v) = (uv, u^2 + v^2, u^2 - v^2).$$

Find the points on the surface corresponding to:

(a) $(u, v) = (1, -1)$.

(b) $(u, v) = (3, 0)$.

(c) $(u, v) = (2, 5)$.

In each case, find x, y, and z by plugging u and v into $S(u, v) = (uv, u^2 + v^2, u^2 - v^2)$.

	(u, v)	(x, y, z)
(a)	$(1, -1)$	$(-1, 2, 0)$
(b)	$(3, 0)$	$(0, 9, 9)$
(c)	$(2, 5)$	$(10, 29, -21)$

□

The next example shows that graphs of functions — but also more general "graphs of functions" can be easily represented parametrically.

Example. (a) Parametrize the graph of $z = x^2 + y^2$.

(b) Parametrize the graph of $x = y^2 + z^2$.

(c) Check that the following equations give a parametrization of $z = x^2 + y^2$:

$$x = u \cos v, \quad y = u \sin v, \quad z = u^2.$$

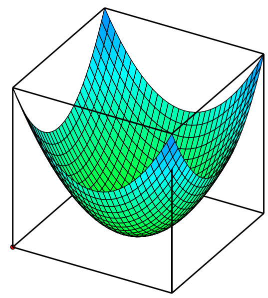
(a) Set each of the independent variables x and y to parameters, so $x = u$ and $y = v$. Then

$$z = x^2 + y^2 = u^2 + v^2.$$

The parametrization is

$$x = u, \quad y = v, \quad z = u^2 + v^2.$$

Here's the graph using this parametrization:



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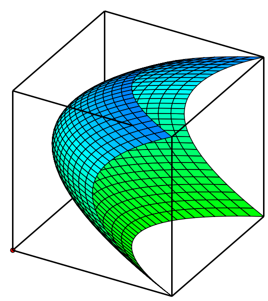
(b) Set each of the independent variables x and y to parameters, so $y = u$ and $z = v$. Then

$$x = y^2 + z^2 = u^2 + v^2.$$

The parametrization is

$$x = u^2 + v^2, \quad y = u, \quad z = v.$$

Here's the graph using this parametrization:



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Notice that the surface is not the graph of a function $z = f(x, y)$, but it's easy to obtain a parametrization. You can see that this procedure will work whenever one of x, y, or z is a function of the other two.

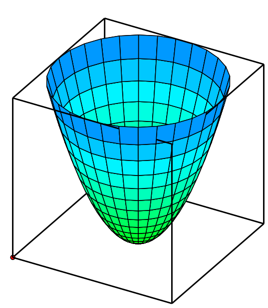
(c) Plug $x = u \cos v$ and $y = u \sin v$ into $z = x^2 + y^2$:

$$z = x^2 + y^2 = (u \cos v)^2 + (u \sin v)^2 = u^2 (\cos v)^2 + u^2 (\sin v)^2 =$$

$$u^2[(\cos v)^2 + (\sin v)^2] = u^2.$$

The z-equation is satisfied.

Here's the graph using this parametrization. Notice that it looks different than the graph in (a).



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The last example shows how you can check that a set of equations parametrizes a surface given in x-y-z form.

Example. The equation of a sphere of radius 1 centered at $(0,0,0)$ is

$$x^2 + y^2 + z^2 = 1.$$

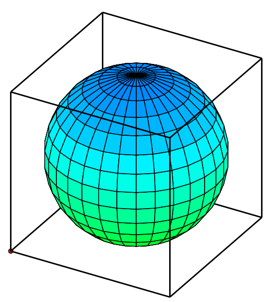
(If "1" is replaced by "a^2" for $a \ge 0$, then the sphere has radius a.) Show that this sphere can be represented parametrically by

$$x = \sin p \cos t, \quad y = \sin p \sin t, \quad z = \cos p.$$

Plug $\sin p \cos t$, $\sin p \sin t$, $\cos p$ into $x^2 + y^2 + z^2 = 1$:

$$\begin{aligned} x^2 + y^2 + z^2 &= (\sin p \cos t)^2 + (\sin p \sin t)^2 + (\cos p)^2 \\ &= (\sin p)^2 (\cos t)^2 + (\sin p)^2 (\sin t)^2 + (\cos p)^2 \\ &= (\sin p)^2 [(\cos t)^2 + (\sin t)^2] + (\cos p)^2 \\ &= (\sin p)^2 + (\cos p)^2 \\ &= 1 \end{aligned}$$

The equation $x^2 + y^2 + z^2 = 1$ is satisfied. Here's the graph using this parametrization:



By the way, the "p" and "t" for the parameters is to anticipate that they eventually "turn into" ρ and θ when we discuss **spherical coordinates**.

Example. Parametrize $z = x^2 - y^2$:

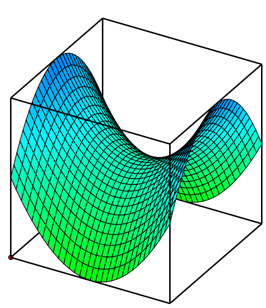
(a) As a function graph.

(b) Using a trig identity. What is a limitation of this method?

(a) Set $x = u$ and $y = v$ and plug into the given equation to get z:

$$x = u, \quad y = v, \quad z = u^2 - v^2.$$

Here's the graph using this parametrization.



(b) The difference of squares makes me think of the identity $(\sec\theta)^2 - (\tan\theta)^2 = 1$. So let

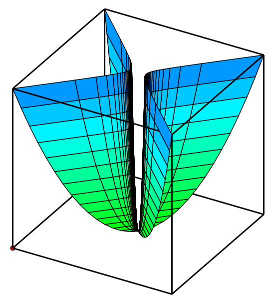
$$x = u\sec v, \quad y = u\tan v.$$

Then

$$z = x^2 - y^2 = (u\sec v)^2 - (u\tan v)^2 = u^2.$$

One problem here is that the range of v must avoid discontinuities in $\sec v$ and $\tan v$: For example, both trig functions "blow up" at $\pm\dfrac{\pi}{2}$. Thus, I have to be a little careful in using this parametrization to graph the surface.

Here's the graph using this parametrization. Notice how different it looks than the graph in (a).



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If a surface is given by an equation involving only two variables, the following procedure can often be used to produce a parametrization.

1. Set the third variable — the one that doesn't appear in the equation — equal to one of the parameters.

2. Think of the given equation as the equation of a curve. Use the remaining parameter to parametrize the curve.

Example. Parametrize the cylinder in $\mathbb{R}^3$ given by

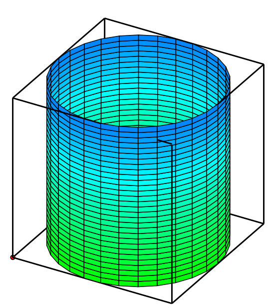
$$x^2 + y^2 = 1.$$

Notice that in 2 dimensions $x^2 + y^2 = 1$ is the equation of a circle. The equation does not involve z, so I set $z = v$.

Next, I must parametrize $x^2 + y^2 = 1$. I can use the standard parametrization of the circle as a curve:

$$x = \cos u, \quad y = \sin u, \quad z = v.$$

Here's the graph using this parametrization.



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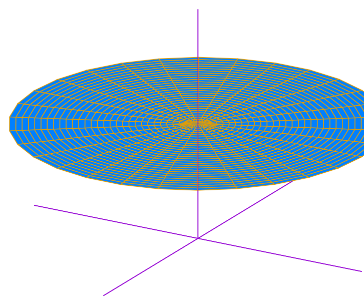
Example. (**A surface lying in a plane**) Describe the surface parametrized by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = 1 \quad \text{for} \quad 0 \le r \le 1 \quad \text{and} \quad 0 \le \theta \le 2\pi.$$

Since $z = 1$, the entire surface lies in the plane $z = 1$.

You may recognize the first two equations as being the polar coordinate conversion equations. For $0 \le r \le 1$ and $\le \theta \le 2\pi$, I will get all the points in the disk of radius 1 centered at the origin — but because $z = 1$, the disk will lie 1 unit above the x-y-plane.

Here's the graph using this parametrization.



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A **surface of revolution** is generated by revolving a curve about a line. The line about which the curve is revolved is called the **axis of revolution**. For simplicity, I'll assume that the line and the curve lie in one of the coordinate planes, and that the axis of revolution is one of the coordinate axes.

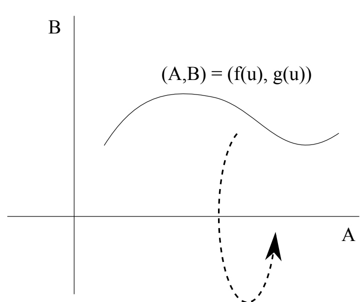
An x-y curve will lie in the x-y-plane, a y-z-curve will lie in the y-z-plane, and an x-z-curve will lie in the x-z-plane. The curve will usually be given in parametric form; if it isn't, you should begin by parametrizing the curve.

To derive a parametrization, I'll use "A", "B", and "C" for the coordinate axes. That way there's less of a risk of confusing variables when I apply these results to a particular problem.

So suppose I have a curve in the A-B plane, and it is being revolved about the A-axis. And suppose the curve has parametric equations

$$A = f(u), \quad B = g(u).$$

I'm writing "A" first and "B" second, but the order isn't important as long as I'm consistent.



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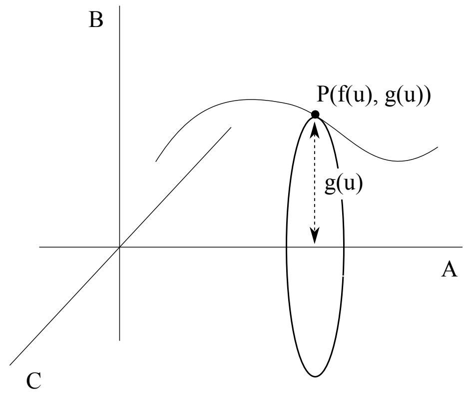
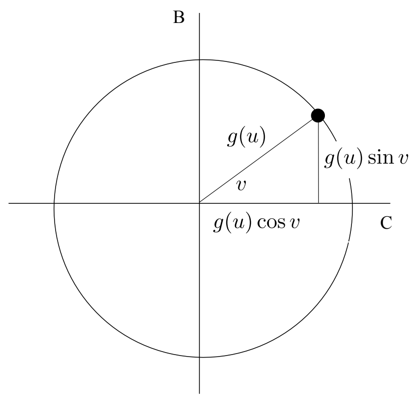
In this picture, you can imagine the C-axis sticking out of the A-B-plane.

A point on the curve moves in a circle around the A-axis. Notice that the circle is parallel to the B-C plane.



The A-coordinate of P is $A = f(u)$. To get the B and C coordinates, look at the circle that P moves in:



Letting v be the radial angle, the right triangle shows that

$$B = g(u)\sin v, \quad C = g(u)\cos v.$$

Therefore, the surface of revolution can be parametrized by

$$A = f(u), \quad B = g(u)\sin v, \quad C = g(u)\cos v.$$

Remarks. (a) One way to think about this is that since the curve is revolved about the A-axis, a point on the surface has the same A-coordinate as the point on the curve that "revolved to it". Thus, $A = f(u)$: The A-component is the same for the curve and the surface.

The revolution is occurring parallel to the B-C-plane, so for B and C you use the B-component for the curve multiplied by $\cos v$ and $\sin v$ to get points on the circle.

(b) If you switch $\cos v$ and $\sin v$ but leave everything else the same, you'll get the same surface. As usual when you do this, the individual circles will get traced out in the opposite direction.

Example. Parametrize the surface generated by revolving the following circle about the z-axis:

$$(y-2)^2 + (z-2)^2 = 1.$$

First, parametrize the circle:

$$y = 2 + \cos u, \quad z = 2 + \sin u.$$

(You can check that it works by substituting for x and y in the circle equation.)

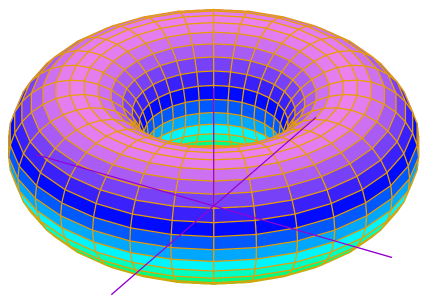
In terms of the notation above, y corresponds to B and z corresponds to A. Hence, x corresponds to C. Thus,

$$f(u) = 2 + \sin u, \quad g(u) = 2 + \cos u.$$

The parametrization is

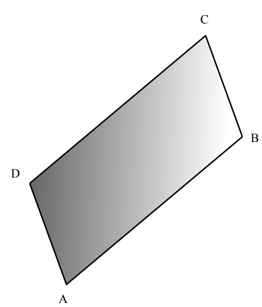
$$x = (2 + \cos u)\cos v, \quad y = (2 + \cos u)\sin v, \quad z = 2 + \sin u.$$

Here's the graph using this parametrization.



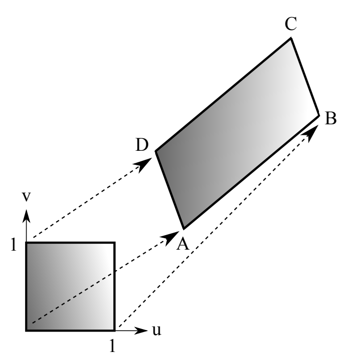
This surface is called a **torus**. $\square$

Next, I'll describe how to parametrize a parallelogram in space. Suppose the vertices of the parallelogram (counterclockwise around the parallelogram) are A, B, C, and D.



I'll construct a function $f : \mathbb{R}^2 \to \mathbb{R}^3$ which maps the unit square $0 \le u \le 1$, $0 \le v \le 1$ to the parallelogram, in such as way that

$$(0,0) \to A, \quad (1,0) \to B, \quad (0,1) \to D.$$



Suppose that A is $A(a_1, a_2, a_3)$ and $\overrightarrow{AB} = (a, b, c)$ and $\overrightarrow{AD} = (d, e, f)$. The transformation is defined by the following matrix equation:

$$(x, y, z) = f(u, v) = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

The matrix $\begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$ deforms the original square so it has the same shape as the parallelogram. Then the vector corresponding to A is added to translate the deformed square onto the parallelogram.

If you need to get equations for x, y, and z, you can multiply out and simplify the right side. It's best to leave the function in matrix form absent a need for the individual equations.

To check that this does what I claimed, try feeding in $(1, 0)$ (for instance):

$$f(1, 0) = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \overrightarrow{AB} + A = B.$$

(For the last equality, note that if you start at A and add the vector from A to B you wind up at B.) You can check in similar fashion that $(0, 0)$ goes to A and $(0, 1)$ goes to D.

The ranges $0 \le u \le 1$ and $0 \le v \le 1$ give the parallelogram. If you use $-\infty < u < \infty$ and $-\infty < v < \infty$ you get the plane containing the parallelogram. This shows how to get parametric equations for a plane.

Example. The vertices of a parallelogram, listed counterclockwise, are $A(1, 1, 2)$, $B(2, 2, 3)$, $C(5, 3, 3)$, and $D(4, 2, 2)$. Parametrize the parallelogram.

The vectors starting at A are

$$\overrightarrow{AB} = (1, 1, 1), \quad \overrightarrow{AD} = (3, 1, 0).$$

The parametrization is

$$(x, y, z) = f(u, v) = \begin{bmatrix} 1 & 3 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

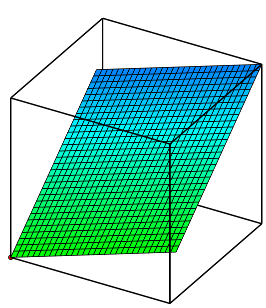
You can write this in component form by multiplying out and simplifying the right side:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u + 3v + 1 \\ u + v + 1 \\ u + 2 \end{bmatrix}.$$

That is,

$$x = u + 3v + 1, \quad y = u + v + 1, \quad z = u + 2.$$

Here's the graph using this parametrization.



$\square$

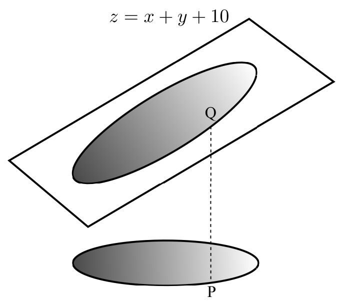
The trick I'll illustrate in the next example works when you can decompose the surface (or its projection into a coordinate plane) into segments. Before I begin, recall that if P and Q are points, the **line segment** from P to Q may be parametrized by

$$(x, y, z) = (1 - u) \cdot P + u \cdot Q, \quad 0 \le u \le 1.$$

If you use $-\infty < u < \infty$, you get the parametric equation for the line through P and Q.

Example. Parametrize the part of the cylinder $x^2 + y^2 = 1$ which extends from the x-y-plane to the plane $z = x + y + 10$.

The following picture illustrates the idea:



For each point P on the circle $x^2 + y^2 = 1$ in the x-y-plane, I construct the segment from P up to the point Q on the plane $z = x + y + 10$. One parameter picks out the point P on the base circle; the other parameter picks out the point on the segment above it.

The circle $x^2 + y^2 = 1$ can be parametrized by

$$x = \cos u, \quad y = \sin u.$$

So $P(\cos u, \sin u, 0)$ is a point on the circle in the x-y-plane. To get the point Q above P in the plane $z = x + y + 10$, substitute $x = \cos u$, $y = \sin u$ in the plane equation:

$$z = \cos u + \sin u + 10.$$

Thus, $Q(\cos u, \sin u, \cos u + \sin u + 10)$ lies above P.

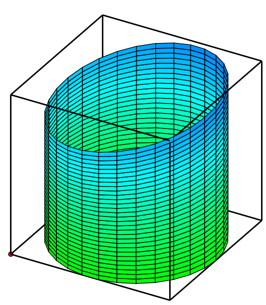
The segment from P to Q is

$$\begin{aligned}(x, y, z) &= (1 - v) \cdot P + v \cdot Q \\ &= (1 - v) \cdot (\cos u, \sin u, 0) + v \cdot (\cos u, \sin u, \cos u + \sin u + 10) \\ &= (\cos u, \sin u, v(\cos u + \sin u + 10)\end{aligned}$$

The parametric equations are

$$x = \cos u, \quad y = \sin u, \quad z = v(\cos u + \sin u + 10).$$

Here's the graph using this parametrization.



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