Path Integrals

A **path integral** in $\mathbb{R}^2$ is the integral of a scalar function $f(x, y)$ along a path $\vec{\sigma}$ in the x-y-plane. Represent the curve in parametrized form: $\sigma(t) = (x(t), y(t))$, or $x = x(t)$, $y = y(t)$ for $a \le t \le b$. Then

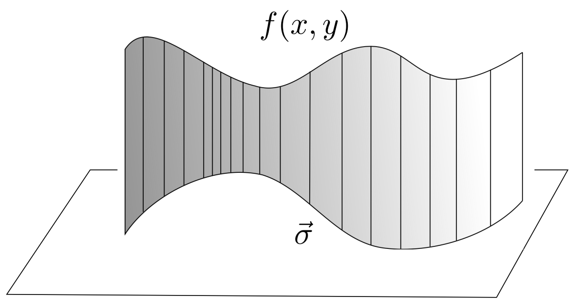
$$\int_{\vec{\sigma}} f\,ds = \int_a^b f(\vec{\sigma}(t))\|\vec{\sigma}\,'(t)\|\,dt.$$

Path integrals in higher dimensions are defined in similar fashion.

Heuristically, the curve is divided into little pieces. A small piece of the curve has length ds, where

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

Above the small piece of the curve, I build a rectangle using f to obtain the height (for example, by plugging a point on the small piece of the curve into f). A careful definition would use Riemann sums, as usual.



It is like building a rectangle sum along a curve, rather than the x-axis as you do with ordinary single-variable integrals.

Example. Compute $\int_C (3x + 2y)\,ds$, where C is the segment from $(1, 3)$ to $(2, -1)$.

The segment from $(1, 3)$ to $(2, -1)$ is

$$(x, y) = (1 - t) \cdot (1, 3) + t \cdot (2, -1) = (1 + t, 3 - 4t).$$

Thus,

$$x = 1 + t, \quad y = 3 - 4t.$$

Hence,

$$ds = \sqrt{x'(t)^2 + y'(t)^2}\,dt = \sqrt{1 + 16}\,dt = \sqrt{17}\,dt.$$

In addition,

$$3x + 2y = 3(1 + t) + 2(3 - 4t) = 9 - 5t.$$

Hence,

$$\int_C (3x + 2y)\,ds = \int_0^1 (9 - 5t)\sqrt{17}\,dt = \sqrt{17}\left[9t - \frac{5}{2}t^2\right]_0^1 = \frac{13\sqrt{17}}{2} = 26.80018\ldots. \quad \square$$

Example. Compute $\displaystyle\int_{\vec{\sigma}} f\,ds$, where $f(x, y) = x^2 - y^2$ and $\vec{\sigma}(t) = (\cos t, \sin t)$, $0 \le t \le \frac{\pi}{4}$.

First, I'll find ds:

$$\vec{\sigma}\,'(t) = (-\sin t, \cos t), \quad \text{so} \quad |\vec{\sigma}\,'(t)| = \sqrt{(\sin t)^2 + (\cos t)^2} = 1.$$

Therefore, $ds = 1 \cdot dt = dt$.
Next, I convert f to a function of t:

$$f(t) = (\cos t)^2 - (\sin t)^2 = \cos 2t.$$

Therefore,

$$\int_{\vec{\sigma}} f\, ds = \int_0^{\pi/4} \cos 2t\, dt = \left[\frac{1}{2}\sin 2t\right]_0^{\pi/4} = \frac{1}{2}. \quad \square$$

Path integrals work in similar fashion in $\mathbb{R}^3$.

Example. Compute $\displaystyle\int_C (x^2 + y + 2z)\, ds$, where C is the segment from $(1, 2, 1)$ to $(2, 0, 1)$.

The segment from $(1, 2, 1)$ to $(2, 0, 1)$ is

$$(x, y, z) = (1 - t) \cdot (1, 2, 1) + t \cdot (2, 0, 1) = (1 + t, 2 - 2t, 1).$$

Hence,

$$ds = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt = \sqrt{1 + 4 + 0}\, dt = \sqrt{5}\, dt.$$

In addition,

$$x^2 + y + 2z = (1 + t)^2 + (2 - 2t) + 2 = t^2 + 5.$$

Therefore,

$$\int_C (x^2 + y + 2z)\, ds = \int_0^1 (t^2 + 5) \cdot \sqrt{5}\, dt = \sqrt{5}\left[\frac{1}{3}t^3 + 5t\right]_0^1 = \frac{16\sqrt{5}}{3} = 11.92569\ldots. \quad \square$$

Example. A wire is bent into the shape of the helix

$$\vec{\sigma}(t) = (\cos t, \sin t, t), \quad 0 \le t \le 4\pi.$$

The density is proportional to the square of the distance from the origin. Find the mass of the wire.

In this case, the curve is 3-dimensional, so I can't picture it as a "fence" as I did with the 2-dimensional curves. However, the computation is essentially the same.
The density is $\delta = k(x^2 + y^2 + z^2)$, where k is a constant. The mass is just $\displaystyle\int_{\vec{\sigma}} \delta\, ds$.
The velocity is

$$\vec{\sigma}\,'(t) = (-\sin t, \cos t, 1), \quad \text{so} \quad |\vec{\sigma}\,'(t)| = \sqrt{(\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}.$$

Hence, $ds = \sqrt{2}\, dt$.
Write δ in terms of t:

$$\delta = k\left[(\cos t)^2 + (\sin t)^2 + 1\right] = k(1 + t^2).$$

The mass is

$$\int_0^{4\pi} k(1 + t^2) \cdot \sqrt{2}\, dt = k\sqrt{2}\left[t + \frac{1}{3}t^3\right]_0^{4\pi} = k\sqrt{2}\left(4\pi + \frac{64}{3}\pi^3\right). \quad \square$$

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