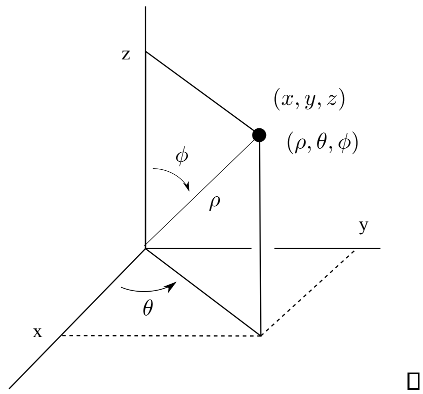
10-14-2023

Spherical Coordinates

Spherical coordinates represent points in $\mathbb{R}^3$ using three numbers: (ρ, θ, ϕ).



ρ is the distance from $(0,0,0)$ to the point.

θ is "the polar coordinate θ" — that is, project the ray from the origin to the point down to a ray $\vec{r}$ in the x-y plane. Measure the angle θ from the positive x-axis to $\vec{r}$ in the usual way.

ϕ is the angle from the positive z-axis to the ray from the origin to the point.

The conversion equations are

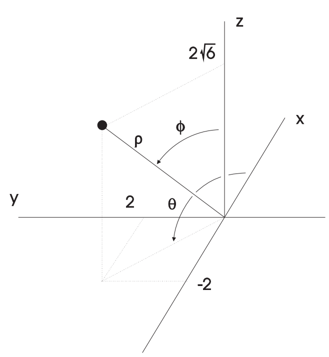
$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi.$$

Note also that

$$\rho^2 = x^2 + y^2 + z^2.$$

However, if you're converting the coordinates of a single point from one coordinate system to another, the best thing is to draw a picture and use trigonometry.

Example. A point has rectangular coordinates $(-2, 2, 2\sqrt{6})$. Find its spherical coordinates.



First, $\theta = \dfrac{3\pi}{4}$. Since $r = 2\sqrt{2}$ and $z = 2\sqrt{6}$, $\rho = 4\sqrt{2}$ by Pythagoras.

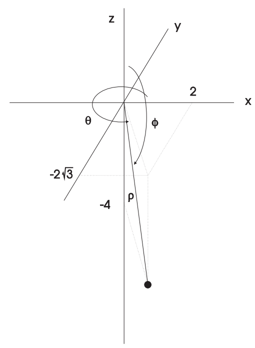
$$\tan\phi = \frac{r}{z} = \frac{2\sqrt{2}}{2\sqrt{6}} = \frac{1}{\sqrt{3}}.$$

So $\phi = \dfrac{\pi}{6}$.

The spherical coordinates are $\left(4\sqrt{2}, \dfrac{3\pi}{4}, \dfrac{\pi}{6}\right)$. □

Note: If a point lies below the x-y-plane, ϕ will be greater than $\dfrac{\pi}{2}$. In that case, you can't use $\sin^{-1}$ or $\tan^{-1}$ "as is" to give ϕ, since those inverse trig functions only produce angles less than or equal to $\dfrac{\pi}{2}$.

Example. A point has cylindrical coordinates $\left(4, \dfrac{5\pi}{3}, -4\right)$. Find its spherical coordinates.



$\theta = \dfrac{5\pi}{3}$. Since $r = 4$ and $z = -4$, $\rho = 4\sqrt{2}$ by Pythagoras. Finally, the radius lies $\dfrac{\pi}{4}$ below the x-y-plane, so $\phi = \dfrac{3\pi}{4}$.

The spherical coordinates are $\left(4\sqrt{2}, \dfrac{5\pi}{3}, \dfrac{3\pi}{4}\right)$. □

Example. Let r be the polar coordinate radius. Express r in terms of spherical coordinates.

$$\begin{aligned} x^2 &= \rho^2(\sin\phi)^2(\cos\theta)^2 \\ y^2 &= \rho^2(\sin\phi)^2(\sin\theta)^2 \\ \hline x^2 + y^2 &= \rho^2(\sin\phi)^2 \\ r^2 &= \rho^2(\sin\phi)^2 \\ r &= \rho\sin\phi \end{aligned}$$

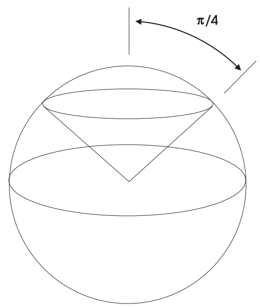
I can take $0 \le \phi \le \pi$ (so $\sin\phi \ge 0$) and $\rho \ge 0$, so I can avoid taking absolute values in the last square root step. □

Example. Sketch the region in space described by the following spherical coordinate inequalities:

$$\left\{ \begin{array}{c} 0 \le \rho \le 1 \\ 0 \le \phi \le \dfrac{\pi}{4} \end{array} \right\}$$

The region lies inside the sphere of radius 1 but above the cone $\phi = \dfrac{\pi}{4}$. Note that the latter is

$$\begin{aligned}
\phi &= \frac{\pi}{4} \\
\tan\phi &= \tan\frac{\pi}{4} = 1 \\
\frac{z}{r} &= 1 \\
z &= r \\
z &= \sqrt{x^2 + y^2}
\end{aligned}$$



□

The spherical conversion equations are

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi.$$

They define a function $f : \mathbb{R}^3 \to \mathbb{R}^3$

$$f(\rho, \theta, \phi) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi).$$

The Jacobian of f is

$$\begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix} = -\rho^2 \sin\phi.$$

The absolute value is $\rho^2 \sin\phi$. Hence, when you go from rectangular coordinates to spherical coordinates, the differentials convert by

$$dx\, dy\, dz \to \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

Therefore, in order to convert a triple integral from rectangular coordinates to spherical coordinates, you should do the following:

1. Convert the limits of integration by describing the region of integration by inequalities in spherical coordinates.

2. Convert the integrand using the spherical conversion formulas:

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi.$$

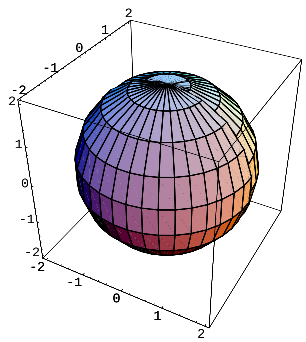
$$\rho^2 = x^2 + y^2 + z^2, \quad \frac{\sqrt{x^2 + y^2}}{z} = \tan\phi.$$

3. Convert the differentials by

$$dx\,dy\,dz \to \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

Example. Let R be the interior of the sphere $x^2 + y^2 + z^2 = 4$. Compute

$$\iiint_R \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}\, dx\, dy\, dz.$$



R is a sphere of radius 2 centered at the origin. The interior of the sphere is described by the inequalities

$$\left\{ \begin{matrix} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi \\ 0 \le \rho \le 2 \end{matrix} \right\}$$

Moreover, since $r = \rho \sin\phi$,

$$\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} = \frac{\rho \sin\phi}{\rho} = \sin\phi.$$

Therefore, the integral becomes

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{2} \rho^2 (\sin\phi)^2\, d\rho\, d\phi\, d\theta = 2\pi \int_0^{\pi} (\sin\phi)^2 \int_0^{2} \rho^2\, d\rho\, d\phi = 2\pi \int_0^{\pi} (\sin\phi)^2 \left[\frac{1}{3}\rho^3\right]_0^2 d\phi =$$

$$\frac{16\pi}{3} \int_0^{\pi} (\sin\phi)^2\, d\phi = \frac{8\pi}{3} \int_0^{\pi} (1 - \cos 2\phi)\, d\phi = \frac{8\pi}{3} \left[\phi - \frac{1}{2}\sin 2\phi\right]_0^{\pi} = \frac{8\pi^2}{3} = 26.31894\ldots. \quad \square$$

When should you consider converting a triple integral to spherical coordinates? Here are two rough guidelines:

(a) Consider converting to spherical coordinates when the region of integration involves graphs that "look nice" in spherical. For example, **spheres** and **cones** often produce regions that can be described by simple inequalities in spherical coordinates.

(b) Consider converting to spherical coordinates when the integrand involves terms like $x^2 + y^2 + z^2$ $(= \rho^2)$.

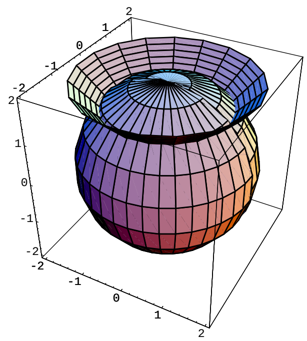
Example. Let R be the region defined by the inequalities $x^2 + y^2 + z^2 \le 4$ and $z \ge \sqrt{x^2+y^2}$. Compute

$$\iiint_R \sqrt{x^2+y^2+z^2}\, dx\, dy\, dz.$$

$x^2+y^2+z^2=4$ is a sphere of radius 2 centered at the origin, so $x^2+y^2+z^2 \le 4$ refers to the interior of the sphere.

$z=\sqrt{x^2+y^2}$ is a cone opening at a 45° angle to the z-axis. Hence, $z \ge \sqrt{x^2+y^2}$ refers to the region lying above the cone.

Together, the inequalities specify the region inside the sphere but above the cone. (It's shaped like an ice-cream cone.)



Evidently, the region "goes all the way around" the z-axis, so $0 \le \theta \le 2\pi$. To determine the inequalities for ρ, think of a searchlight beam emanating from the origin. The beam *enters* the ice-cream cone at the origin ($\rho = 0$) and *leaves* the ice-cream cone through the top of the ice-cream, which is our sphere ($\rho = 2$). Hence, $0 \le \rho \le 2$.

ϕ is the angle measured "downward" from the z-axis. Think of an umbrella held upside-down. If you open the umbrella, through what range of angles will the ribs sweep as they pass through the region? Since the cone makes a 45° ($=\dfrac{\pi}{4}$) angle with the z-axis, it follows that $0 \le \phi \le \dfrac{\pi}{4}$.

The inequalities in spherical coordinates which describe the region are

$$\left\{\begin{matrix} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \dfrac{\pi}{4} \\ 0 \le \rho \le 2 \end{matrix}\right\}$$

Finally, since $\sqrt{x^2+y^2+z^2} = \rho$, the integral is

$$\int\int\int_R \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz = \int_0^{2\pi}\int_0^{\pi/4}\int_0^2 \rho^3 \sin\phi\,d\rho\,d\phi\,d\theta = 2\pi \int_0^{\pi/4} \sin\phi \left[\frac{1}{4}\rho^4\right]_0^2 d\phi =$$

$$8\pi \int_0^{\pi/4} \sin\phi\,d\phi = 8\pi \left[-\cos\phi\right]_0^{\pi/4} = 8\pi\left(1 - \frac{\sqrt{2}}{2}\right) = 7.36120\ldots. \quad \square$$
