4-2-2018

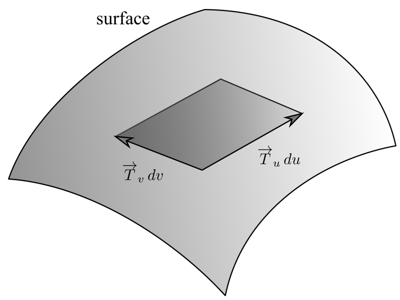
Surface Area

I'll consider the problem of finding the area of part of a surface in $\mathbb{R}^3$:

$$x = f(u, v), \quad y = g(u, v), \quad z = h(u, v).$$

Here is a heuristic motivation for the formula.

In our discussion of the tangent plane to a surface, we found a normal vector by taking two vectors in the tangent plane and taking their cross product. We considered a small piece of the surface near the point of tangency. A small piece will be nearly flat, and will look like the parallelogram depicted below (exaggerated so you can see it):



The sides of the parallelogram are determined by the tangent vectors $\vec{T}_u$ and $\vec{T}_v$, scaled up the the parameter increments du and dv. The area of the parallelogram should be the length of the cross product:

$$\|\vec{T}_v\, du \times \vec{T}_v\, dv\| = \|\vec{T}_v \times \vec{T}_v\|\, du\, dv.$$

To get the area of the part of the surface corresponding to the region R in the u-v plane, integrate to add up the areas of the parallelograms:

$$\iint_R \|\vec{T}_u \times \vec{T}_v\|\, du\, dv.$$

As a special case, consider a surface $z = f(x, y)$ given as the graph of a function. A normal vector is given by

$$\vec{N} = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right).$$

Hence,

$$\|\vec{N}\| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}.$$

In this case, the area of the surface is

$$\iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dx\, dy.$$

Example. Find the area of the part of the surface $z = 2xy - 1$ which lies inside the cylinder $x^2 + y^2 = 9$.

$$\frac{\partial z}{\partial x} = 2y, \quad \frac{\partial z}{\partial y} = 2x, \quad \text{so} \quad \|\vec{N}\| = \sqrt{4y^2 + 4x^2 + 1}.$$

The region of integration is the interior of the circle $x^2 + y^2 = 9$:

$$\left\{\begin{array}{c} -3 \le x \le 3 \\ -\sqrt{9 - x^2} \le y \le \sqrt{9 - x^2} \end{array}\right\}$$

I'll convert to polar. The region becomes

$$\left\{\begin{array}{c} 0 \le \theta \le 2\pi \\ 0 \le r \le 3 \end{array}\right\}$$

The integrand is

$$\|\vec{N}\| = \sqrt{4r^2 + 1}.$$

The area is

$$\int_0^3 \int_0^{2\pi} \sqrt{4r^2 + 1} r \, d\theta \, dr = 2\pi \int_0^3 \sqrt{4r^2 + 1} r \, dr = 2\pi \left[\frac{1}{12}(4r^2 + 1)^{3/2}\right]_0^3 = \frac{\pi}{6}(37^{3/2} - 1) = 117.31870\ldots. \quad \square$$

Example. Find the area of the surface

$$x = v \cos u, \quad y = v \sin u, \quad z = v, \quad 0 \le u \le 2\pi, \quad 0 \le v \le 2.$$

$$\vec{T}_u = (-v \sin u, v \cos u, 0), \quad \vec{T}_v = (\cos u, \sin u, 1),$$

$$\vec{T}_u \times \vec{T}_v = \left|\begin{array}{ccc} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -v \sin u & v \cos u & 0 \\ \cos u & \sin u & 1 \end{array}\right| = (v \cos u, v \sin u, -v),$$

$$\|\vec{T}_u \times \vec{T}_v\| = \sqrt{v^2(\cos u)^2 + v^2(\sin u)^2 + v^2} = \sqrt{2}v.$$

The area of the surface is

$$\int_0^2 \int_0^{2\pi} \sqrt{2}v \, du \, dv = 2\pi\sqrt{2} \int_0^2 v \, dv = 2\pi\sqrt{2} \left[\frac{1}{2}v^2\right]_0^2 = 4\pi\sqrt{2} = 17.77153\ldots. \quad \square$$

Example. (a) Find the area of the sphere $x^2 + y^2 + z^2 = a^2$ by representing the top hemisphere as the graph of a function.

(b) Find the area of the sphere $x^2 + y^2 + z^2 = a^2$ using the parametrization

$$x = a \cos u \cos v, \quad y = a \cos u \sin v, \quad z = a \sin u, \quad -\frac{\pi}{2} \le u \le \frac{\pi}{2}, \quad 0 \le v \le 2\pi.$$

(a) The top hemisphere is

$$z = \sqrt{a^2 - x^2 - y^2}.$$

Thus,

$$\vec{N} = \left(-\frac{x}{\sqrt{a^2 - x^2 - y^2}}, -\frac{y}{\sqrt{a^2 - x^2 - y^2}}, -1\right),$$

$$\|\vec{N}\| = \left(\frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2} + 1\right)^{1/2} = \left(\frac{a^2}{a^2 - x^2 - y^2}\right)^{1/2} = \frac{a}{\sqrt{a^2 - x^2 - y^2}}.$$

The region of integration is the interior of the circle $x^2 + y^2 = a^2$:

$$\begin{gathered} -a \le x \le a \\ -\sqrt{a^2 - x^2} \le y \le \sqrt{a^2 - x^2} \end{gathered}$$

I'll convert to polar. The region is

$$\begin{gathered} 0 \le \theta \le 2\pi \\ 0 \le r \le a \end{gathered}$$

Likewise,

$$\|\vec{N}\| = \frac{a}{\sqrt{a^2 - r^2}}.$$

I need to double the area for the top hemisphere to get the area of the whole sphere:

$$2 \int_0^a \int_0^{2\pi} \frac{a}{\sqrt{a^2 - r^2}} r \, d\theta \, dr = 4\pi \int_0^a \frac{ar}{\sqrt{a^2 - r^2}} \, dr = 4\pi a \left[-\sqrt{a^2 - r^2}\right]_0^a = 4\pi a^2. \quad \Box$$

(b)

$$\vec{T}_u = (-a \sin u \cos v, -a \sin u \sin v, a \cos u), \quad \vec{T}_v = (-a \cos u \sin v, a \cos u \cos v, 0),$$

$$\vec{T}_u \times \vec{T}_v = \left|\begin{array}{ccc} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -a \sin u \cos v & -a \sin u \sin v & a \cos u \\ -a \cos u \sin v & a \cos u \cos v & 0 \end{array}\right| = (-a^2 (\cos u)^2 \cos v, -a^2 (\cos u)^2 \sin v, -a^2 \sin u \cos u),$$

$$\|\vec{T}_u \times \vec{T}_v\| = \sqrt{a^4 (\cos u)^4 (\cos v)^2 + a^4 (\cos u)^4 (\sin v)^2 + a^4 (\sin u)^2 (\cos u)^2} = a^2 \cos u.$$

The area is

$$\int_{-\pi/2}^{\pi/2} \int_0^{2\pi} a^2 \cos u \, dv \, du = 2\pi a^2 \int_{-\pi/2}^{\pi/2} \cos u \, du = 2\pi a^2 \left[\sin u\right]_{-\pi/2}^{\pi/2} = 4\pi a^2. \quad \Box$$
