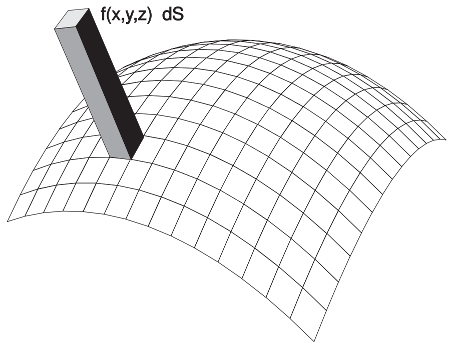
4-24-2018

Surface Integrals

If S is a surface and $f(x, y, z)$ is a function, the **scalar surface integral** of f over S is

$$\iint_S f\, dS.$$

Imagine placing a grid on the surface. dS represents the area of a small parallelogram in the grid. At a point (x, y, z), build a "box" on the grid at (x, y, z) whose height is $f(x, y, z)$. The volume of the box will be product of the height $(f(x, y, z))$ and the parallelogram area (dS), i.e. $f(x, y, z)dS$.



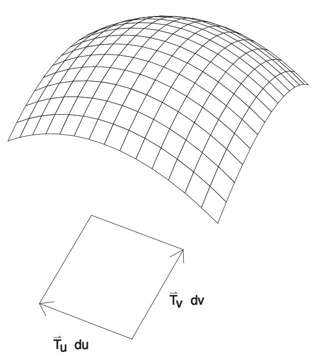
Under this heuristic interpretation, the scalar surface integral represents the total volume of all the "boxes" built in this way on the surface.

It is fairly clear how to deal with $f(x, y, z)$, but what about dS? Suppose the surface is parametrized:

$$(x, y, z) = \Phi(u, v).$$

That is, each of x, y, and z is expressed in terms of parameters u and v. Fix v and vary u. This gives a curve on the surface, whose tangent vector I'll denote by $\vec{T}_u$. Likewise, fixing u and varying v produces a curve on the surface whose tangent vector I'll denote by $\vec{T}_v$.

A small part of the surface grid pictured above can be thought of as a parallelogram whose sides are given by the vectors $\vec{T}_u\, du$ and $\vec{T}_v\, dv$.



$\vec{T}_u$ gives the "rate of change" of S with respect to u; multiplying by a small change du in u gives the approximate change $\vec{T}_u\, du$ in S. Likewise, $\vec{T}_v\, dv$ gives the change in S produced by a small change in v.

The area of the little parallelogram is the length of the cross product of its sides:

$$dS = \|\vec{T}_u \times \vec{T}_v\|\, du\, dv.$$

$\vec{T}_u \times \vec{T}_v$ is the **normal vector** to the surface, since each factor is tangent to the surface.

Thus, to compute a scalar surface integral, use

$$\iint_S f\,dS = \iint_D f\left(x(u,v), y(u,v), z(u,v)\right) \|\vec{T}_u \times \vec{T}_v\|\,du\,dv.$$

The region D which gives the bounds for the double integral is given by the ranges for the parameters u and v.

"$f\left(x(u,v), y(u,v), z(u,v)\right)$" means that you should use the parametric equations for the surface to convert f from x, y, and z to u and v.

You can compute the normal vector using

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix}.$$

If the surface is given as the graph of a function $z = f(x,y)$, you'll integrate over the projection D of the surface into the x-y plane.

You will probably do the integral using x and y as the variables, but you might want to convert to polar coordinates if the double integral warrants it.

Finally, a normal is given by

$$\vec{N} = \pm\left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right).$$

Hence,

$$\|\vec{N}\| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}.$$

Example. Let S be the part of the plane $z = 2x + 5y + 1$ lying above the square

$$\left\{\begin{matrix} 0 \le x \le 1 \\ 0 \le y \le 1 \end{matrix}\right\}$$

Let $f(x,y,z) = 3x + y - z$. Compute $\displaystyle\iint_S f\,dS$.

The normal vector to the plane is

$$\vec{N} = (-2, -5, 1), \quad \text{so} \quad \|\vec{N}\| = \sqrt{4 + 25 + 1} = \sqrt{30}.$$

I have

$$f(x,y,z) = 3x + y - z = 3x + y - (2x + 5y + 1) = x - 4y - 1.$$

Hence,

$$\iint_S f\,dS = \int_0^1 \int_0^1 (x - 4y - 1)\sqrt{30}\,dx\,dy = \sqrt{30}\int_0^1 \left[\frac{1}{2}x^2 - 4xy - x\right]_0^1 dy = \sqrt{30}\int_0^1 \left(-\frac{1}{2} - 4y\right) dy =$$

$$\sqrt{30}\left[-\frac{1}{2}y - 2y^2\right]_0^1 = -\frac{5\sqrt{30}}{2} = -13.69306\ldots. \quad \square$$

If f represents the density of a sheet of material having the form of the surface S, then the surface integral $\iint_S f\, dS$ gives the mass of the sheet.

Example. A sheet of metal of varying density has the form of the surface

$$x = u\cos v, \quad y = u\sin v, \quad z = u^3, \quad 0 \le u \le 1, \quad 0 \le v \le 2\pi.$$

Suppose the density is $\delta(x,y,z) = x^2 + y^2$. Find the mass of the sheet of metal.

$$\vec{T}_u = (\cos v, \sin v, 3u^2), \quad \vec{T}_v = (-u\sin v, u\cos v, 0).$$

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos v & \sin v & 3u^2 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = (-3u^3\cos v, -3u^3\sin v, u).$$

$$\|\vec{T}_u \times \vec{T}_v\| = \sqrt{9u^6(\cos v)^2 + 9u^6(\sin v)^2 + u^2} = \sqrt{9u^6 + u^2} = u\sqrt{9u^4+1}.$$

$$\delta(x,y,z) = x^2 + y^2 = u^2(\cos v)^2 + u^2(\sin v)^2 = u^2.$$

Hence, the mass is

$$\int_0^{2\pi}\int_0^1 u^2 \cdot u\sqrt{9u^4+1}\, du\, dv = 2\pi\int_0^1 u^3\sqrt{9u^4+1}\, du = 2\pi\int_1^{10} u^3\sqrt{w}\cdot\frac{dw}{36u^3} =$$

$$\left[w = 9u^4 + 1, \quad dw = 36u^3\, du, \quad du = \frac{dw}{36u^3}; \quad u = 0, w = 1; u = 1, w = 10\right]$$

$$\frac{\pi}{18}\int_1^{10}\sqrt{w}\, dw = \frac{\pi}{18}\left[\frac{2}{3}w^{3/2}\right]_1^{10} = \frac{\pi}{27}\left(10^{3/2} - 1\right) = 3.56312\ldots. \quad \square$$

Now consider a vector field $\vec{F}$ in space, and let S be a surface. If you think of F as the velocity field of a fluid or gas and the surface S as a membrane, it is natural to ask "how much" fluid or gas passes through the membrane per unit time. This rate is called the **flux** of $\vec{F}$ through S, and is given by the **vector surface integral**

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(\vec{T}_u \times \vec{T}_v)\, du\, dv.$$

(I'm assuming that the surface is parametrized by $(x,y,z) = \Phi(u,v)$.)
If the surface is given as the graph of a function $z = f(x,y)$, a normal is given by

$$\vec{N} = \pm\left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right).$$

You must decide whether to use $\vec{N}$ or $-\vec{N}$ based on the wording of the problem.

Example. Let S be the part of the surface $z = x^2 + y^2$ lying below the plane $z = 4$. Find the flux of $\vec{F} = (x, y, -5z)$ upward through S.

$$\vec{N} = \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right) = (-2x, -2y, 1).$$

Note that this normal vector has positive z-component, which is correct for computing the flux *upward* through S.

Then

$$\vec{F} = (x, y, -5z) = (x, y, -5(x^2 + y^2)).$$

So

$$\vec{F} \cdot \vec{N} = -2x^2 - 2y^2 - 5(x^2 + y^2) = -7(x^2 + y^2).$$

I'll do the double integral in polar.

$z = x^2 + y^2$ intersects $z = 4$ in $x^2 + y^2 = 4$, so the projection into the x-y-plane is

$$\left\{ \begin{matrix} 0 \le \theta \le 2\pi \\ 0 \le r \le 2 \end{matrix} \right\}$$

And

$$\vec{F} \cdot \vec{N} = -7r^2.$$

So the flux is

$$\int_0^{2\pi} \int_0^2 -7r^2 \cdot r\,dr\,d\theta = -14\pi \int_0^2 r^3\,dr = -14\pi \left[\frac{1}{4}r^4\right]_0^2 = -56\pi = -175.92918\ldots. \quad \square$$

Example. Compute the flux of $\vec{F} = (x, y, -z)$ upward through the surface

$$x = u + 2v, \quad y = 2u + v, \quad z = 3uv, \quad 0 \le u \le 1, \quad 0 \le v \le 1.$$

$$\vec{T}_u = (1, 2, 3v) \quad \text{and} \quad \vec{T}_v = (2, 1, 3u).$$

A normal vector is

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3v \\ 2 & 1 & 3u \end{vmatrix} = (6u - 3v, -3u + 6v, -3).$$

However, I want to compute the flux *upward* through the surface, and this normal has *negative* z-component. So I use the negative of this normal vector, which is

$$-(\vec{T}_u \times \vec{T}_v) = (-6u + 3v, 3u - 6v, 3).$$

Next,

$$\vec{F} = (u + 2v, 2u + v, -3uv).$$

So

$$\vec{F} \cdot [-(\vec{T}_u \times \vec{T}_v)] = (-6u^2 - 9uv + 6v^2) + (6u^2 - 9uv - 6v^2) - 9uv = -27uv.$$

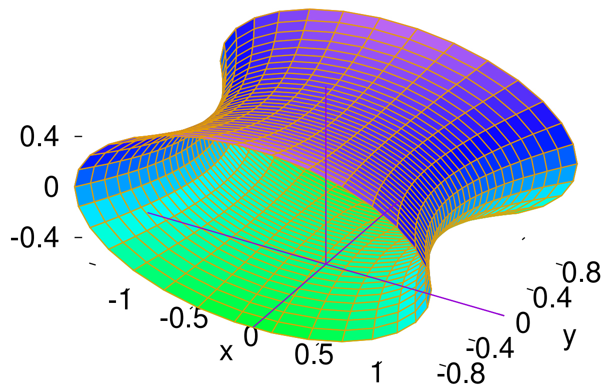
The flux is

$$\int_0^1 \int_0^1 -27uv\,du\,dv = -27 \int_0^1 \left[\frac{1}{2}u^2 v\right]_0^1 dv = -\frac{27}{2} \int_0^1 v\,dv =$$

$$-\frac{27}{2} \left[\frac{1}{2}v^2\right]_0^1 = -\frac{27}{4}. \quad \square$$

Example. The elliptic hyperboloid $x^2 - y^2 + 4z^2 = 1$ may be parametrized by

$$x = \sec u \cos v, \quad y = \tan u, \quad z = \frac{1}{2} \sec u \sin v.$$



Compute the flux of the radial vector field $\vec{F} = (x, y, z)$ outward through the part of the surface in determined by the parameter ranges

$$\left\{ \begin{array}{c} -\dfrac{\pi}{4} \le u \le \dfrac{\pi}{4} \\ 0 \le v \le 2\pi \end{array} \right\}$$

The normal is

$$\vec{T}_u \times \vec{T}_v = \left| \begin{array}{ccc} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \sec u \tan u \cos v & (\sec u)^2 & \dfrac{1}{2} \sec u \tan u \sin v \\ -\sec u \sin v & 0 & \dfrac{1}{2} \sec u \cos v \end{array} \right| =$$

$$\left(\frac{1}{2} (\sec u)^3 \cos v, -\frac{1}{2} (\sec u)^2 \tan u, (\sec u)^3 \sin v \right).$$

For the given ranges of u and v, the x and z components of the normal are *positive*, so the normal points *out of* the hyperboloid. (If the normal had turned out to point *inward*, I'd have simply multiplied it by -1 to get the outward normal.)

Next, write the field in terms of u and v:

$$\vec{F} = \left(\sec u \cos v, \tan u, \frac{1}{2} \sec u \sin v \right).$$

Therefore,

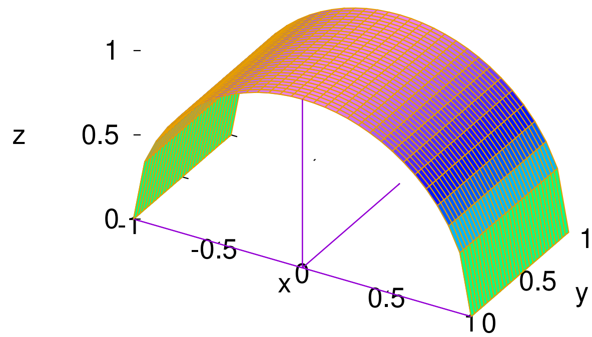
$$\vec{F} \cdot (\vec{T}_u \times \vec{T}_v) = \frac{1}{2} (\sec u)^2.$$

Hence, the flux is

$$\int\int_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_{-\pi/4}^{\pi/4} \frac{1}{2} (\sec u)^2 \, du \, dv = \pi \, [\tan u] - \pi/4^{\pi/4} = 2\pi = 6.28319. \quad \square$$

Example. Find the flux of $\vec{F} = (2x, 0, z)$ out of the part of the cylinder $z = \sqrt{1 - x^2}$ lying above the region

$$\left\{ \begin{array}{c} -1 \le x \le 1 \\ 0 \le y \le 1 \end{array} \right\}$$



The normal is

$$\left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right) = \left(\frac{x}{\sqrt{1-x^2}}, 0, 1\right).$$

This cylinder is an "x-z" cylinder, with the y-axis as its axis. So the *inward* normal will have *negative* x and z components, while the *outward* normal will have *positive* x and z components. The normal above has positive x and z components, so it's the right one.

Next,

$$\vec{F} \cdot \vec{N} = (2x, 0, z) \cdot \left(\frac{x}{\sqrt{1-x^2}}, 0, 1\right) = \frac{1+x^2}{\sqrt{1-x^2}}.$$

Hence, the flux is

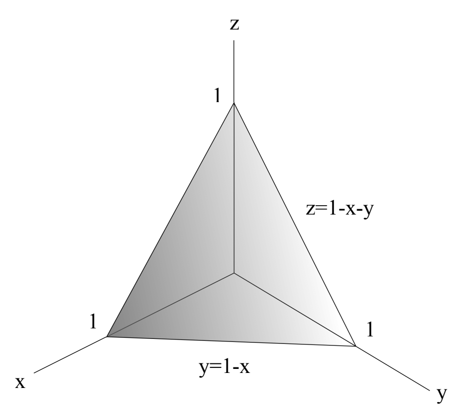
$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^1 \int_{-1}^1 \frac{1+x^2}{\sqrt{1-x^2}}\, dx\, dy = \frac{3\pi}{2} = 4.71238\ldots. \quad \square$$

In the next problem, the vector surface integral is given in a form like the differential form of a line integral.

Example. Let S be the part of the plane $x + y + z = 1$ lying in the first quadrant.

Compute

$$\iint_S xz\, dx\, dy + xy\, dy\, dz + yz\, dx\, dz.$$



I will do each of the terms separately. First, if R_{xy} is the projection of the surface into the x-y plane,

$$\iint_S xz\, dx\, dy = \iint_{R_{xy}} xz\, dx\, dy.$$

(I project into the x-y plane because the differentials are $dx\,dy$.) Now $z = 1 - x - y$, and the projection is

$$\left\{ \begin{matrix} 0 \le x \le 1 \\ 0 \le y \le 1 - x \end{matrix} \right\}$$

So

$$\iint_{R_{xy}} xz\,dx\,dy = \int_0^1 \int_0^{1-x} x(1-x-y)\,dy\,dx = \int_0^1 x\left[y - xy - \frac{1}{2}y^2\right]_0^{1-x} dx =$$

$$\int_0^1 x\left((1-x) - x(1-x) - \frac{1}{2}(1-x)^2\right) dx = \int_0^1 \left(\frac{1}{2}x^3 - x^2 + \frac{1}{2}x\right) dx =$$

$$\left[\frac{1}{8}x^4 - \frac{1}{3}x^3 + \frac{1}{4}x^2\right]_0^1 = \frac{1}{24}.$$

Similarly, if R_{yz} and R_{xz} are the projections into the y-z and x-z planes, respectively, then

$$\iint_S xy\,dy\,dz = \int_0^1 \int_0^{1-y} (1-y-z)y\,dz\,dy = \frac{1}{24}$$

$$\iint_S yz\,dx\,dz = \int_0^1 \int_0^{1-x} (1-x-z)z\,dz\,dx = \frac{1}{24}$$

The total is $\dfrac{1}{24} + \dfrac{1}{24} + \dfrac{1}{24} = \dfrac{1}{8}$. $\square$
