Surfaces

A **surface** is a 2-dimensional object in $\mathbb{R}^3$. Later I'll discuss **parametric surfaces** and give a more precise definition. This section is primarily descriptive, to acquaint you with some of the objects we'll use in multivariable calculus.

Just as the graph of a function $y = f(x)$ is a curve in $\mathbb{R}^2$, the graph of a function $z = f(x, y)$ is a surface in $\mathbb{R}^3$. In principle, you could draw the graph of $z = f(x, y)$ by plotting points, as you can for the graph of $y = f(x)$.

Example. Consider the function $z = x^2 + xy$.

(a) Is the point $(2, 1, 6)$ on the graph of f?

(b) Is the point $(1, -3, 2)$ on the graph of f?

(c) Complete the following table, filling in the z-values:

x	y	$f(x, y) = x^2 + xy$
2	0	
-1	4	
0	5	
1	2	

To determine whether a point (x, y, z) is on the graph, plug x and y into $f(x, y)$ and see if you get z.

(a) $f(2, 1) = 2^2 + 2 \cdot 1 = 6$, so the point is on the graph. $\square$

(b) $f(1, -3) = 1^2 + 1 \cdot (-3) = -2 \neq 2$, so the point is not on the graph. $\square$

(c)

x	y	$f(x, y) = x^2 + xy$
2	0	4
-1	4	-3
0	5	0
1	2	3

$\square$

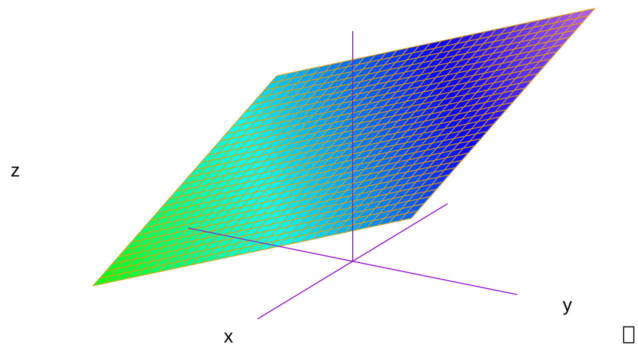
The last example shows the limitation of drawing surfaces by plotting points: With 2 input variables (compared to one input variable for $y = f(x)$) it is even more tedious than drawing curves by plotting points. For that reason, I'll usually assume that you have access to computer software that can draw the graphs for you. Try to plot the surfaces in the examples yourself using a software package like *Mathematica* or *maxima*. Some of the graphs that occur a lot (like planes, spheres, cylinders, cones, or paraboloids) you will probably learn to sketch by hand.

Example. Sketch the plane $4x + 3y - z = 18$.

You could do this by finding the intercepts of the plane with the x, y, and z axes. Alternatively, solve for z:

$$z = 4x + 3y - 18.$$

Now I can use software to draw the graph:



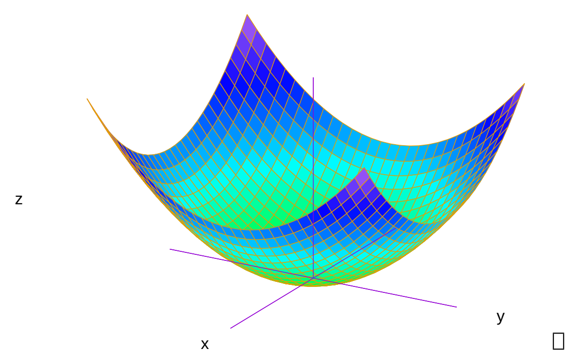
Example. Sketch the paraboloid:

(a) $z = x^2 + y^2 + 6$.

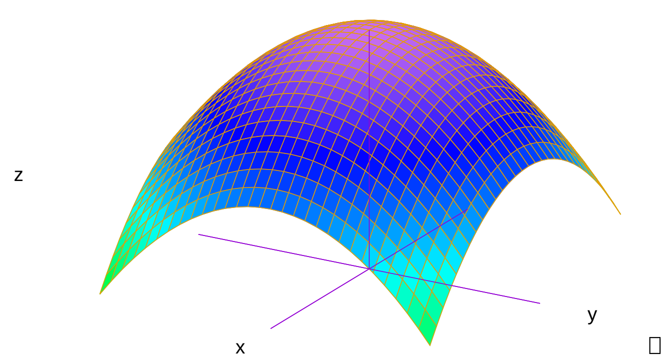
(b) $z = 7 - x^2 - y^2$.

A **paraboloid** is the surface-analog of a parabola. In fact, you can get a paraboloid by revolving a parabola about its axis.

(a)



(b)



Notice how the form of the equation relates to whether the paraboloid opens upward or downward.

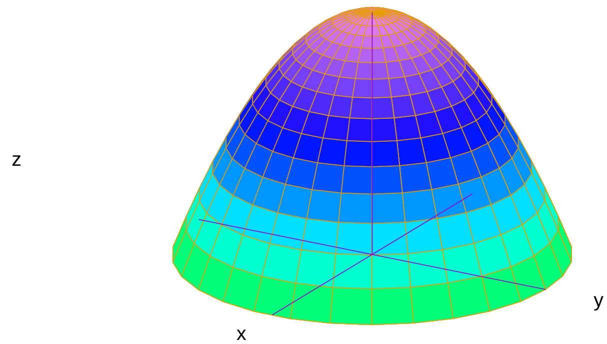
Example. Find and describe the curve of intersection of the paraboloid $z = 4 - x^2 - y^2$ and the x-y-plane.

The x-y-plane has the equation $z = 0$. Setting $z = 0$ in the paraboloid equation, I get

$$0 = 4 - x^2 - y^2$$
$$x^2 + y^2 = 4$$

This is a circle of radius 2 centered at the origin in the x-y-plane. □

Here's a picture:



Notice that this paraboloid is "round" compared to the earlier paraboloids. To get this picture (so you could see that the intersection is a circle) I needed to represent the paraboloid **parametrically**. I'll discuss parametric surfaces later, but if you want to try this for yourself, use:

$$x = u \cos v, \quad y = u \sin v, \quad z = 4 - u^2.$$

As in the last example, I'll describe some of the surfaces below **parametrically** and discuss parametrizing surfaces in more detail later. You'll probably be able to figure out (a bit) how parametric surfaces work by following along with the examples.

Just as some curves in the planes are not the graphs of functions because they fail the "vertical line test", some useful surfaces are not graph of functions (because they fail the "vertical line test" for functions of 2 variables). In this case, the failure of the test means that for some pairs of inputs (x, y), there are multiples z-values satisfying the equation. You can draw some of these surfaces by drawing them "in pieces".

Example. Graph the sphere

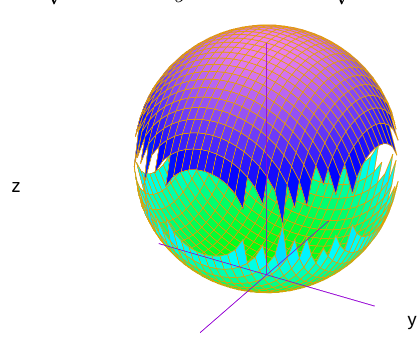
$$x^2 + y^2 + z^2 = 9.$$

Notice what happens if I solve for z:

$$z = \pm\sqrt{9 - x^2 - y^2}.$$

I actually get two functions, one for the plus sign and one for the minus sign. The plus sign gives the top hemisphere and the minus sign gives the bottom hemisphere.

If I graph the two functions $z = \sqrt{9 - x^2 - y^2}$ and $z = -\sqrt{9 - x^2 - y^2}$ together, I get:



□

Note: I could graph the sphere "all at once" by representing it **parametrically**. I'll discuss parametric surfaces later, but if you want to try this for yourself, use:

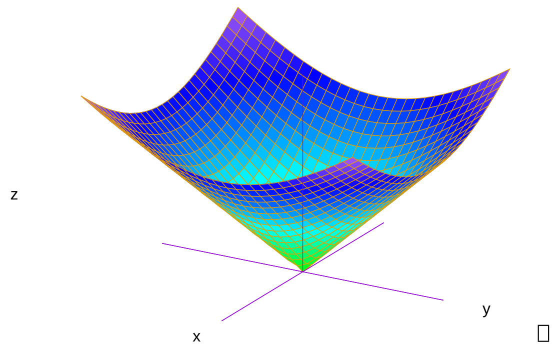
$$x = 3\cos u \cos v, \quad y = 3\cos u \sin v, \quad z = 3\sin u.$$

Example. Graph the cones:

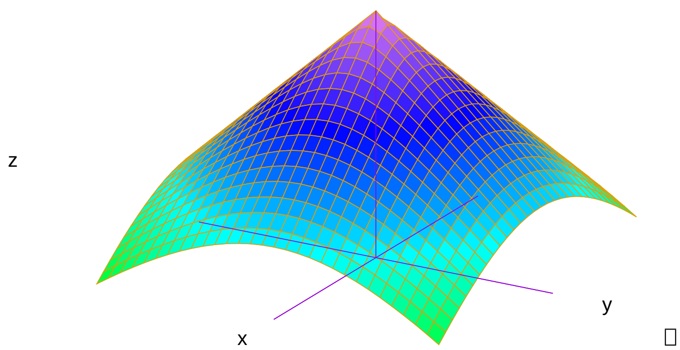
(a) $z = \sqrt{x^2 + y^2} + 2$.

(b) $z = 1 - \sqrt{x^2 + x^2}$.

(a)



(b)

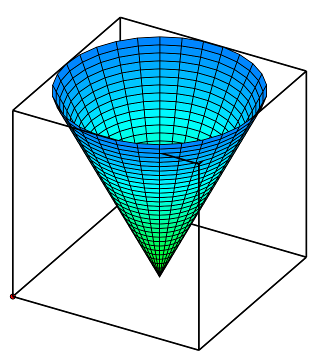


These graphs may not look like cones to you. If you looked at the last few examples, you might anticipate that I could get better pictures by representing the surfaces **parametrically**.

For the first cone, use:

$$x = u\cos v, \quad y = u\sin v, z = u + 2.$$

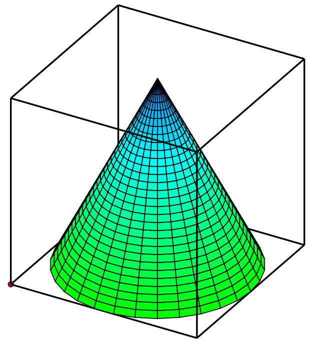
The graph looks like this:



For the second cone, use:

$$x = u\cos v, \quad y = u\sin v, z = 1 - u.$$

The graph looks like this:

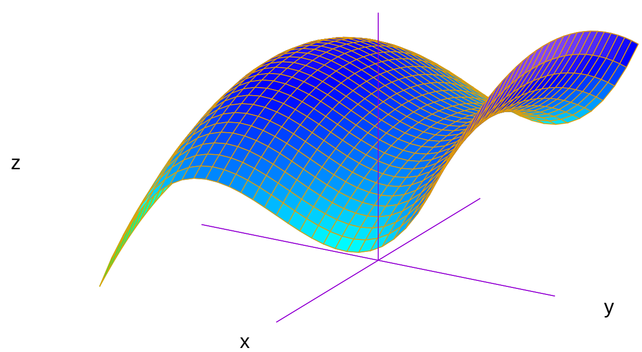


Those look more like cones!

After seeing the last few examples, you might be anxious to move on to parametric surfaces. Still, it's worth remembering that you can obtain many useful surfaces as graphs of functions.

Example. Graph $z = x^3 - 4 * x - y^2$.

If I draw the graph "as-is", I get:

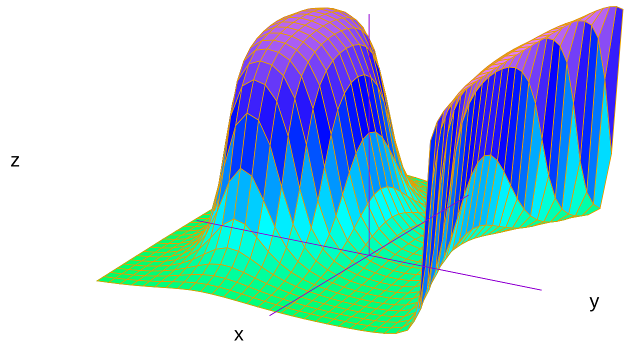


However, this graph may be too "flat" for you to be able to discern some important features of the graph. If you don't care about scale and you want to exaggerate the features of a surface, you can wrap $f(x, y)$ in $\tan^{-1}$, so you end up plotting $\tan^{-1} f(x, y)$.

(I first saw this trick in Stan Wagon's book [1], which contains lots of neat projects. It is written for *Mathematica*, but you can translate the projects to other computer algebra systems like *maxima* without too much trouble.)

Why does this work? Remember that the inverse tangent function takes $(-\infty, \infty)$ to $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$. So taking $\tan^{-1}$ has the effect of compressing an infinite range down to a finite range, and exaggerating features which would otherwise be "smeared out".

Here's how the surface $z = \tan^{-1}(x^3 - 4 * x - y^2)$ looks:



Now you can clearly see a a **local maximum** (the "mountain") with a **saddle point** to its right. □

Example. Sketch the cylinder:

(a) $x^2 + y^2 = 1$.

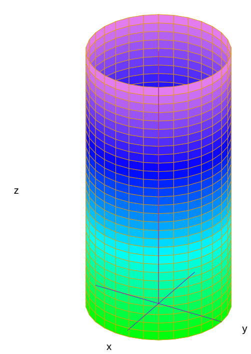
(b) $x^2 + z^2 = 1$.

(c) $y^2 + z^2 = 1$.

In all of these cases, I used a **parametric representation** for the cylinder, simply because it's easier to obtain the graphs.

(a) For $x^2 + y^2 = 1$, I used

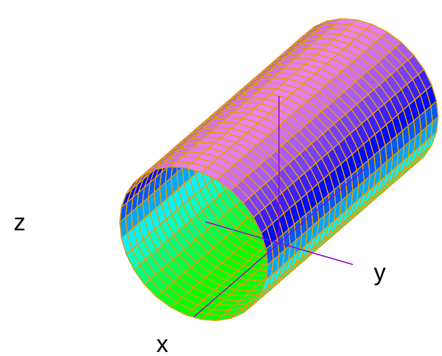
$$x = \cos u, \quad y = \sin u, \quad z = v.$$



□

(b) For $x^2 + z^2 = 1$, I used

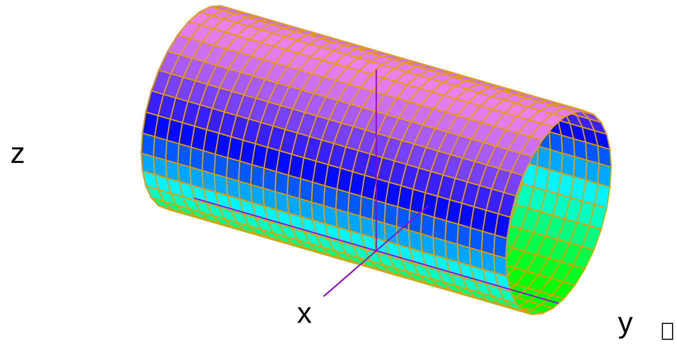
$$x = \cos u, \quad y = v, \quad z = \sin u.$$



□

(c) For $x^2 + z^2 = 1$, I used

$$x = v, \quad y = \cos u, \quad z = \sin u.$$



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[1] Stan Wagon, *Mathematica in action.* New York: W. H. Freeman and Company, 1991. (See pages 88–89 for the inverse tangent trick)