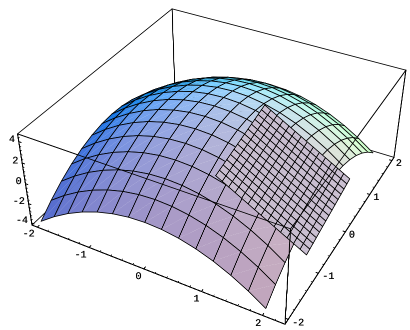
The Tangent Plane to a Surface

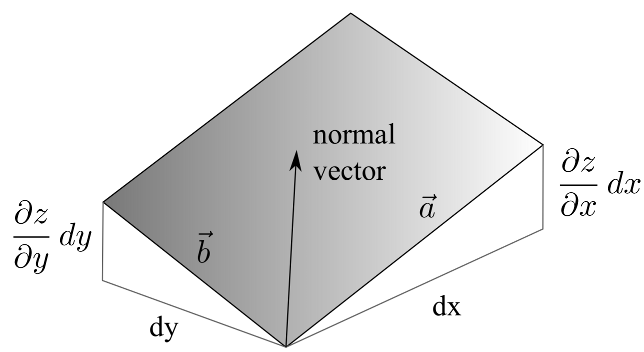
The derivative of a function of one variable gives the slope of the tangent line to the graph. The partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ of a function of two variables $z = f(x, y)$ determine the **tangent plane** to the graph.



The graph of $z = f(x, y)$ is a surface in 3 dimensions. Suppose we're trying to find the equation of the tangent plane at $(a, b, f(a, b))$.

To write down the equation of a plane, we need a point on the plane and a vector perpendicular to the plane. We have a point on the plane, namely $(a, b, f(a, b))$.

To find a vector perpendicular to the plane, we find two vectors in the plane and take their cross product. To do this, look at a small piece of the surface near the point of tangency. A small piece will be nearly flat, and will look like the parallelogram depicted below:



The vectors $\vec{a}$ and $\vec{b}$ which are the sides of the parallelogram are tangent to the surface at the point of tangency. Consider $\vec{a}$. It runs in the x-direction. A small change dx in x produces a change in z — the amount the vector $\vec{a}$ "rises".

How much does z change due to a change dx in x? The rate of change of z with respect to x is $\dfrac{\partial z}{\partial x}$, so the change in z produced by changing x by dx is just $\dfrac{\partial z}{\partial x} dx$.

Now $\vec{a}$ is a vector with x-component dx, no y-component, and z-component $\dfrac{\partial z}{\partial x} dx$. Therefore,

$$\vec{a} = \left(dx, 0, \frac{\partial z}{\partial x} dx\right).$$

A similar argument shows that

$$\vec{b} = \left(0, dy, \frac{\partial z}{\partial y} dy\right).$$

The cross product is

$$\vec{a} \times \vec{b} = \left(-\frac{\partial z}{\partial x} dx\, dy, -\frac{\partial z}{\partial y} dx\, dy, dx\, dy\right) = \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right) dx\, dy.$$

I need *any* vector perpendicular to the surface. Since vectors which are multiples are parallel, I may use this vector as the perpendicular vector to the surface:

$$\left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right).$$

This is often referred to as the **normal vector** to the surface and denoted by $\vec{N}$.

The tangent plane at $(a, b, f(a, b))$ is

$$\left(-\left.\frac{\partial f}{\partial x}\right|_{(a,b)}\right)(x - a) + \left(-\left.\frac{\partial f}{\partial y}\right|_{(a,b)}\right)(y - b) + (z - f(a, b)) = 0.$$

The normal line to the surface at $(a, b, f(a, b))$ is the line which passes through $(a, b, f(a, b))$ and is perpendicular to the tangent plane. The normal line is parallel to the normal vector $\left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right)$. Therefore, the parametric equations of the normal line are

$$x - a = -\left.\frac{\partial f}{\partial x}\right|_{(a,b)} \cdot t, \quad y - b = -\left.\frac{\partial f}{\partial y}\right|_{(a,b)} \cdot t, \quad z - f(a, b) = t.$$

Example. Find the equation of the tangent plane and the parametric equations of the normal line to $z = \frac{2x}{y} - x^2$ at $(1, 1, 1)$.

The normal vector to the surface is

$$\left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right) = \left(-\frac{2}{y} + 2x, \frac{2x}{y^2}, 1\right).$$

Plugging in $x = 1$ and $y = 1$ gives $(0, 2, 1)$.

The tangent plane is

$$0 \cdot (x - 1) + 2 \cdot (y - 1) + 1 \cdot (z - 1) = 0, \quad \text{or} \quad 2y + z = 3.$$

The normal line is

$$x - 1 = 0, \quad y - 1 = 2t, \quad z - 1 = t. \quad \square$$

Example. Use a tangent plane to approximate $(1.99)^2 - \frac{1.99}{1.01}$.

The idea is to think of this as the result of plugging numbers into a function $z = f(x, y)$. What is f? Well, the form of the expression suggests that 1.99 corresponds to one of the variables and 1.01 to the other. It's natural to use the function

$$z = f(x, y) = x^2 - \frac{x}{y}.$$

I want to approximate $f(1.99, 1.01)$. The point $(1.99, 1.01)$ is close to $(2, 1)$, so I'll use the tangent plane at $(2, 1)$ to approximate f.

The normal vector is

$$\left(-2x + \frac{1}{y}, -\frac{x}{y^2}, 1\right).$$

Plug in $x = 2$, $y = 1$. This gives $(-3, -2, 1)$.
When $x = 2$ and $y = 1$, $z = 2$. The point of tangency is $(2, 1, 2)$.
The tangent plane is

$$-3(x-2) - 2(y-1) + (z-2) = 0, \quad \text{or} \quad z = 3x + 2y - 6.$$

Now set $x = 1.99$, $y = 1.01$. This gives $z \approx 1.99$. (The actual value is 1.989803.)

Here is an equivalent way to think of things that is similar to the "approximation by differentials" technique you may have seen in first-year calculus. The change Δf in f produced by small changes in dx in x and dy in y is approximated by

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy.$$

Thus,

$$f(x + dx, y + dy) \approx f(x, y) + df.$$

Here (x, y) denotes the "nice" point ($(2, 1)$ in the last example) and $(x + dx, y + dy)$ denotes the "ugly" point ($(1.99, 1.01)$ in the last example).
If you redo the example using this differential approach, you'd have

$$f(1.99, 1.01) \approx f(2, 1) + \left(\frac{\partial f}{\partial x}\right)(dx) + \left(\frac{\partial f}{\partial y}\right)(dy) = 2 + (3)(-0.01) + (2)(0.01) = 1.99. \quad \square$$

Suppose a surface is given parametrically:

$$x = f(u, v), \quad y = g(u, v), \quad z = h(u, v).$$

Consider the vectors

$$\vec{T}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) \quad \text{and} \quad \vec{T}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right).$$

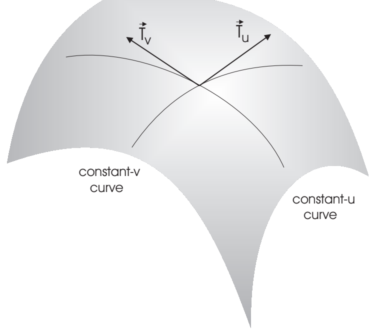
These vectors are tangent to the *curves in the surface* determined by letting one of u or v vary and holding the other constant. For example, if u varies and $v = c$ is constant, I get the curve

$$x = f(u, c), \quad y = g(u, c), \quad z = h(u, c).$$

The velocity vector for this curve is $\vec{T}_u$.
Likewise, consider the curve obtained by setting u to a constant:

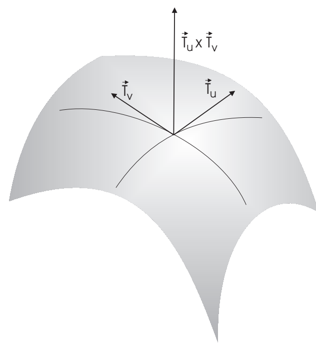
$$x = f(c, v), \quad y = g(c, v), \quad z = h(c, v).$$

The velocity vector for this curve is $\vec{T}_v$.



The cross product of $\vec{T}_u$ and $\vec{T}_v$ is a normal vector to the surface:

$$N = \vec{T}_u \times \vec{T}_v.$$



Example. Find the equation of the tangent plane and the parametric equations for the normal line to

$$x = u^2 - v^2, \quad y = uv, \quad z = u^2 + v^2 \quad \text{at} \quad (u, v) = (2, 1).$$

First, the point of tangency is obtained by plugging $u = 2$ and $v = 1$ into x, y, and z. I get $x = 3$, $y = 2$, and $z = 5$. The point is $(3, 2, 5)$.

Next,

$$\vec{T}_u = (2u, v, 2u) \quad \text{and} \quad \vec{T}_v = (-2v, u, 2v).$$

When $u = 2$ and $v = 1$,

$$\vec{T}_u = (4, 1, 4) \quad \text{and} \quad \vec{T}_v = (-2, 2, 2).$$

The normal vector is

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 1 & 4 \\ -2 & 2 & 2 \end{vmatrix} = (-6, -16, 10).$$

The tangent plane is

$$-6(x - 3) - 16(y - 2) + 10(z - 5) = 0.$$

The normal line is

$$x - 3 = -6t, \quad y - 2 = -16t, \quad z - 5 = 10t. \quad \square$$
