Taylor Series for Functions of Several Variables

You've seen Taylor series for functions $y = f(x)$ of 1 variable. For a function $f : \mathbb{R} \to \mathbb{R}$ satisfying the appropriate conditions, we have

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x,c).$$

$R_n(x,c)$ is the **remainder term**:

$$R_n(x,c) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}.$$

z is a number between c and x. The Remainder Term gives the error that occurs in approximating $f(x)$ with the n^{th} degree Taylor polynomial.

There is a similar formula for functions of several variables. To make the notation a little better, I'll define **higher-order differentials** as follows. Let $h = (h_1, h_2, \ldots, h_n) \in \mathbb{R}^n$.

$$D^2 f(x,h) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} \cdot h_i h_j.$$

$$D^3 f(x,h) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k} \cdot h_i h_j h_k.$$

And so on. Here's Taylor's formula for functions of several variables. With more variables, it's more complicated and technical; try to see the resemblance between the formula here and the one for functions of one variable.

Theorem. Suppose $f : U \to \mathbb{R}$, where U is an open set in $\mathbb{R}^n$. Suppose f has continuous partial derivatives at all points of U through order $m+1$. Let $x, c \in U$, where $x \neq c$ and the segment from c to x is contained in U. Then for some point z on the segment from c to x,

$$f(x) = f(c) + \sum_{k=1}^{m} \frac{1}{k!} D^k f(c, x-c) + \frac{1}{(m+1)!} D^{(m+1)} f(z, x-c). \quad \square$$

Example. Write out the Taylor expansion through terms of degree 2 for a function of 2 variables $z = f(x,y)$.

Let's say we're expanding at a point (c,d). Then

$$f(x,y) = f(c,d) + \left(\frac{\partial f}{\partial x}(c,d) \cdot (x-c) + \frac{\partial f}{\partial y}(c,d) \cdot (y-d) \right) +$$

$$\frac{1}{2!} \left(\frac{\partial^2 f}{\partial x^2}(c,d)\ (x-c)^2 + 2 \frac{\partial^2 f}{\partial x \partial y}(c,d)\ (x-c)(y-d) + \frac{\partial^2 f}{\partial y^2}(c,d)\ (y-d)^2 \right) + \cdots. \quad \square$$

Example. For a function $z = f(x,y)$,

$$f(1,2) = 29, \quad f_x(1,2) = 19, \quad f_y(1,2) = 24,$$

$$f_{xx}(1,2) = 22, \quad f_{xy}(1,2) = 8, \quad f_{yy}(1,2) = 10.$$

Write out the Taylor expansion of f at $(1,2)$ through terms of degree 2.

$$f(x,y) = 29 + 19(x-1) + 24(y-2) + \frac{1}{2}\left(22(x-1)^2 + 16(x-1)(y-2) + 10(y-2)^2\right) + \cdots. \quad \Box$$

Example. Construct the Taylor series through the 2^{nd} order for $f(x,y) = x^2y + y^2$ at $(x,y) = (1,3)$.

$$f(1,3) = 12.$$

$$\frac{\partial f}{\partial x} = 2xy, \quad \frac{\partial f}{\partial x}(1,3) = 6.$$

$$\frac{\partial f}{\partial y} = x^2 + 2y, \quad \frac{\partial f}{\partial y}(1,3) = 7.$$

$$\frac{\partial^2 f}{\partial x^2} = 2y, \quad \frac{\partial^2 f}{\partial x^2}(1,3) = 2.$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2x, \quad \frac{\partial^2 f}{\partial x \partial y}(1,3) = 2.$$

$$\frac{\partial^2 f}{\partial y^2} = 2, \quad \frac{\partial^2 f}{\partial y^2}(1,3) = 2.$$

$$f(x,y) = 12 + 6(x-1) + 7(y-3) + \frac{1}{2}\left(2(x-1)^2 + 4(x-1)(y-3) + 2(y-3)^2\right) + \cdots. \quad \Box$$

Example. Let $f(x,y) = x\sqrt{y}$. Use a 1^{st}-order Taylor approximation to approximate $f(5.9, 4.1)$.

I'll use a Taylor expansion at $(6,4)$, since it's the closest "nice" point to $(5.9, 4.1)$.

$$f(6,4) = 12.$$

$$\frac{\partial f}{\partial x} = \sqrt{y}, \quad \frac{\partial f}{\partial x}(6,4) = 2.$$

$$\frac{\partial f}{\partial y} = \frac{x}{2\sqrt{y}}, \quad \frac{\partial f}{\partial y}(6,4) = \frac{3}{2}.$$

The series is

$$f(x,y) = 12 + 2(x-6) + \frac{3}{2}(y-4) + \cdots.$$

Then

$$f(5.9, 4.1) \approx 12 + 2(5.9-6) + \frac{3}{2}(4.1-4) = 11.95. \quad \Box$$
