Vector Functions

A function $f : \mathbb{R} \to \mathbb{R}^n$ is called a **vector function** in $\mathbb{R}^n$. We'll focus on vector functions in the plane $\mathbb{R}^2$ and space $\mathbb{R}^3$, but everything goes over to $\mathbb{R}^n$ without any difficulty.

Example. Find $f(0)$ and $f(2)$ for $f : \mathbb{R} \to \mathbb{R}^3$ given by

$$f(t) = (t^2 + 6, \sin \pi t, \ln(t + 2)).$$

$$f(0) = (6, 0, \ln 2) \quad \text{and} \quad f(2) = (10, 0, \ln 4). \quad \square$$

It is no accident that the function in the last example looked like a parametric curve, because that's what it is. A vector function in $\mathbb{R}^n$ is a curve in $\mathbb{R}^n$.

Our main concern is the calculus of vector functions, and the basic idea is that we do everything component-by-component, and so many of the things you learned in single-variable calculus carry over with just minor adjustments.

Definition. Suppose $f : \mathbb{R} \to \mathbb{R}^n$ is a vector function, $c \in \mathbb{R}$, and $L \in \mathbb{R}^n$. Then $\lim\limits_{t\to c} f(t) = L$ means: For every $\epsilon > 0$, there is a δ, such that

$$\delta > |t - c| > 0 \quad \text{implies} \quad \epsilon > \|f(t) - L\|.$$

("$\|f(t) - L\|$" means the length of $f(t) - f(c)$, regarded as a vector in $\mathbb{R}^n$.)

Proposition. Suppose $f : \mathbb{R} \to \mathbb{R}^n$ has components

$$f(t) = (f_1(t), f_2(t), \ldots f_n(t)).$$

Then

$$\lim_{t\to c} f(t) = (\lim_{t\to c} f_1(t), \lim_{t\to c} f_2(t), \ldots \lim_{t\to c} f_n(t)).$$

This means that the limit on the left exists if and only if all the component limits on the right exist, and in that case the two sides are equal. $\square$

In other words, you take the limit of a vector function by taking the limit of each component. All of the usual rules for computing limits work, one component at a time.

Example. Find the value of $\lim\limits_{t\to 1} \left(\dfrac{t^2 + 3t - 4}{t - 1}, \cos \dfrac{\pi t}{2}, 4e^{t-1}\right)$ if it exists.

The component functions of $f(t) = \left(\dfrac{t^2 + 3t - 4}{t - 1}, \cos \dfrac{\pi t}{2}, 4e^{t-1}\right)$ are $f_1(t) = \dfrac{t^2 + 3t - 4}{t - 1}$, $f_2(t) = \cos \dfrac{\pi t}{2}$, and $f_3(t) = 4e^{t-1}$. You can see that the component functions are ordinary one-variable functions of the kind you see in a first-term calculus course.

Note that

$$\lim_{t\to 1} \frac{t^2 + 3t - 4}{t - 1} = \lim_{t\to 1} \frac{(t + 4)(t - 1)}{t - 1} = \lim_{t\to 1}(t + 4) = 5.$$

So

$$\lim_{t\to 1} \left(\frac{t^2 + 3t - 4}{t - 1}, \cos \frac{\pi t}{2}, 4e^{t-1}\right) = (5, 0, 4). \quad \square$$

Definition. Let $f : \mathbb{R} \to \mathbb{R}^n$ be a vector function in $\mathbb{R}^n$, and let $c \in \mathbb{R}$. Then f is continuous at c if $\lim_{t \to c} f(t) = f(c)$.

Proposition. Suppose $f : \mathbb{R} \to \mathbb{R}^n$ has components

$$f(t) = (f_1(t), f_2(t), \ldots f_n(t)).$$

Then f is continuous at c if and only if f_1, f_2, $\ldots f_n$ are continuous at c, considered as functions $\mathbb{R} \to \mathbb{R}$. □

If that looks a bit technical, don't worry. The meaning is that a vector function is continuous at a point if its component functions are.

Example. Define $f : \mathbb{R} \to \mathbb{R}^2$ by

$$f(t) = \begin{cases} (t, t^2 + 3) & \text{if } t \neq 0 \\ (0, 2) & \text{if } t = 0 \end{cases}$$

Prove or disprove: f is continuous at $t = 0$.

$$\lim_{t \to 0} f(t) = \lim_{t \to 0} (t, t^2 + 3) = (0, 3), \quad \text{but} \quad f(0) = (0, 2).$$

Since $\lim_{t \to 0} f(t) \neq f(0)$, the function is not continuous at $t = 0$. □

Definition. Let $f : \mathbb{R} \to \mathbb{R}^n$ be a vector function in $\mathbb{R}^n$. The **derivative** of f is the vector function $f' : \mathbb{R} \to \mathbb{R}^n$ given by

$$f'(t) = (f_1'(t), f_2'(t), \ldots f_n'(t)).$$

I'll often write $\dfrac{df(t)}{dt}$ or $\dfrac{df}{dt}$ for $f'(t)$.

Proposition. Let $f, g : \mathbb{R} \to \mathbb{R}^n$ be vector functions in $\mathbb{R}^n$, and let $c \in \mathbb{R}$. Then:

(a) If $c = (c_1, c_2, \ldots c_n)$ is a constant, then $\dfrac{dc}{dt} = \vec{0}$.

(b) $\dfrac{d}{dt}[f(t) + g(t)] = \dfrac{df}{dt} + \dfrac{dg}{dt}$.

(c) $\dfrac{d}{dt}(c \cdot f(t)) = c \cdot \dfrac{df}{dt}$.

(d) (**Dot product**) $\dfrac{d}{dt}(f(t) \cdot (g(t)) = \dfrac{df}{dt} \cdot g(t) + f(t) \cdot \dfrac{dg}{dt}$.

Note: In (d), all the products are dot products.

(e) (**Cross product**) Suppose $n = 3$. Then

$$\frac{d}{dt}(f(t) \times g(t)) = \frac{df}{dt} \times g(t) + f(t) \times \frac{dg}{dt}.$$

Proof. The proofs amount to proving the results component-wise. For example, consider (d). Suppose

$$f(t) = (f_1(t), f_2(t), \ldots f_n(t)) \quad \text{and} \quad g(t) = (g_1(t), g_2(t), \ldots g_n(t)).$$

Then

$$f(t) \cdot g(t) = f_1(t)g_1(t) + f_2(t)g_2(t) + \cdots + f_n(t)g_n(t).$$

Using the Product Rule for functions of one variable, I have

$$\begin{aligned}
\frac{d}{dt} f(t) \cdot g(t) &= \frac{d}{dt}[f_1(t)g_1(t) + f_2(t)g_2(t) + \cdots + f_n(t)g_n(t)] \\
&= [f_1'(t)g_1(t) + f_1(t)g_1'(t)] + [f_2'(t)g_2(t) + f_2(t)g_2'(t)] + \cdots + [f_n'(t)g_n(t) + f_n(t)g_n'(t)] \\
&= [f_1'(t)g_1(t) + f_2'(t)g_2(t) + \cdots + f_n'(t)g_n(t)] + [f_1(t)g_1'(t) + f_2(t)g_2'(t) + \cdots + f_n(t)g_n'(t)] \\
&= (f_1'(t), f_2'(t), \ldots f_n'(t)) \cdot (g_1(t), g_2(t), \ldots g_n(t)) + (f_1(t), f_2(t), \ldots f_n(t)) \cdot (g_1'(t), g_2'(t), \ldots g_n'(t)) \\
&= \frac{df}{dt} \cdot g(t) + f(t) \cdot \frac{dg}{dt}
\end{aligned}$$

The other results are proved in similar fashion. □

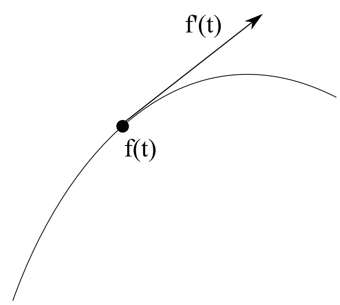
Example. Let

$$f(t) = (t^2 + 3t + 1, 17, \sin 4t).$$

Compute $f'(t)$ and $f'(1)$.

$$f'(t) = (2t + 3, 0, 4\cos 4t) \quad \text{and} \quad f'(1) = (5, 0, 4\cos 4). \quad □$$

Thinking of $f : \mathbb{R} \to \mathbb{R}^n$ as a curve, $f'(t)$ is a **tangent vector** to the curve.



Example. Find parametric equations for the tangent line to

$$f(t) = (t^3 + 5, (t + 1)^2, 7t + 1) \quad \text{at} \quad t = 1.$$

The point of tangency is $f(1) = (6, 4, 8)$. Now

$$f'(t) = (3t^2, 2(t + 1), 7) \quad \text{so} \quad f'(1) = (3, 4, 7).$$

Thus, $(3, 4, 7)$ is a vector tangent to the curve, so it's parallel to the tangent line to the curve. The tangent line is

$$x - 6 = 3t, \quad y - 4 = 4t, \quad z - 8 = 7t. \quad □$$

You can integrate vector functions component-by-component.

Definition. Suppose $f : \mathbb{R} \to \mathbb{R}^n$ has components

$$f(t) = (f_1(t), f_2(t), \ldots f_n(t)).$$

Then

$$\int f(t)\,dt = \left(\int f_1(t)\,dt, \int f_2(t)\,dt, \ldots \int f_n(t)\,dt\right).$$

A similar definition holds for definite integrals.

Proposition. Let $f, g : \mathbb{R} \to \mathbb{R}^n$ be vector functions in $\mathbb{R}^n$, and let $c \in \mathbb{R}$. Then:

(a) $\displaystyle\int [f(t) + g(t)]\,dt = \int f(t)\,dt + \int g(t)\,dt.$

(b) $\displaystyle\int cf(t)\,dt = c\int f(t)\,dt.$ $\square$

Example. Compute the integral $\displaystyle\int (4 - (\sec t)^2, e^{6t}, 12t^2 - 8t + 5)\,dt.$

$$\int (4 - (\sec t)^2, e^{6t}, 12t^2 - 8t + 5)\,dt = \left(4t - \tan t, \frac{1}{6}e^{6t}, 4t^3 - 4t^2 + 5t\right) + (c_1, c_2, c_3). \quad \square$$

Example. Compute the integral $\displaystyle\int_0^1 \left(6t^2 + 5, \frac{3t+1}{,}6\cos 3t\right)\,dt.$

$$\int_0^1 \left(6t^2 + 5, \frac{3t+1}{,}6\cos 3t\right)\,dt = \left[\left(2t^3 + 5t, \frac{1}{3}\ln|3t+1|, 2\sin 3t\right)\right]_0^1 = \left(7, \frac{1}{3}\ln 4, 2\sin 3\right). \quad \square$$
