1-23-2018

Vectors

I'll look at **vectors** from an *algebraic* point of view and a *geometric* point of view.

Algebraically, a **vector** is an ordered list of (usually) real numbers. Here are some 2-dimensional vectors:

$$(2,-3), \quad \left(\sqrt{2}, \frac{3}{5}\right), \quad (0,0).$$

The numbers which make up the vector are the vector's **components**.

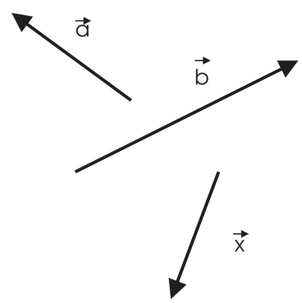
Here are some 3-dimensional vectors:

$$(1,2,-17), \quad \left(\frac{1}{\sqrt{2}}, -17, \pi\right), \quad (0,0,0).$$

Since we usually use x, y, and z as the coordinate variables in 3 dimensions, a vector's components are sometimes referred to as its x, y, and z-components. For instance, the vector $(1,2,-17)$ has x-component 1, it has y-component 2, and it has z-component -17.

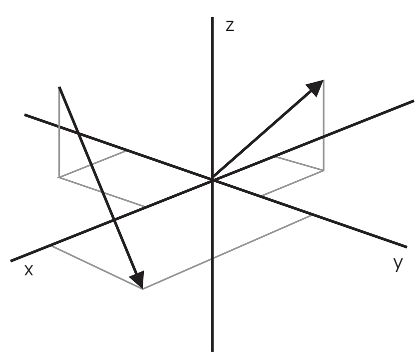
The set of 2-dimensional real-number vectors is denoted $\mathbb{R}^2$, just like the set of ordered pairs of real numbers. Likewise, the set of 3-dimensional real-number vectors is denoted $\mathbb{R}^3$.

Geometrically, a vector is represented by an arrow. Here are some 2-dimensional vectors:



A vector is commonly denoted by putting an arrow above its symbol, as in the picture above.

Here are some 3-dimensional vectors:

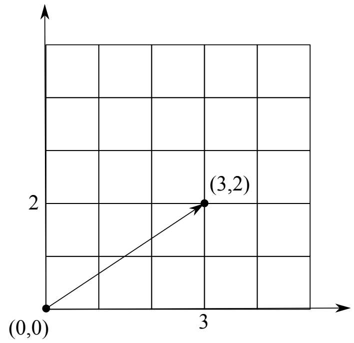


The relationship between the algebraic and geometric descriptions comes from the following fact: The vector from a point $P(a,b)$ to a point $Q(c,d)$ is given by $\overrightarrow{PQ} = (c-a, d-b)$.

In 3 dimensions, the vector from a point $P(a,b,c)$ to a point $Q(d,e,f)$ is $\overrightarrow{PQ} = (d-a, e-b, f-c)$.

Remark. You've probably already noticed the following harmless confusion: "$(3,2)$" can denote the **point** $(3,2)$ in the x-y-plane, or the 2-dimensional real vector $(3,2)$. Notice that the vector from the origin $(0,0)$

to the point $(3,2)$ is the vector $(3,2)$.



So we can usually regard them as interchangeable. When there's a need to make a distinction, I will call it out.

Example. (a) Find the vector from $P(3,4,-7)$ to $Q(-2,2,5)$.

(b) Find the vectors $\overrightarrow{AB}$, $\overrightarrow{BA}$, $\overrightarrow{AC}$, and $\overrightarrow{CD}$ for the points $A(1,1)$, $B(2,3)$, $C(-2,0)$, and $D(-1,2)$. Sketch the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

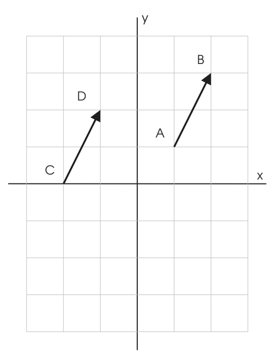
(a)

$$\overrightarrow{PQ} = (-2-3, 2-4, 5-(-7)) = (-5,-2,12). \quad \square$$

(b)

$$\overrightarrow{AB} = (2-1, 3-1) = (1,2),$$
$$\overrightarrow{BA} = (1-2, 1-3) = (-1,-2),$$
$$\overrightarrow{AC} = (-2-1, 0-1) = (-3,-1),$$
$$\overrightarrow{CD} = (-1-(-2), 2-0) = (1,2).$$

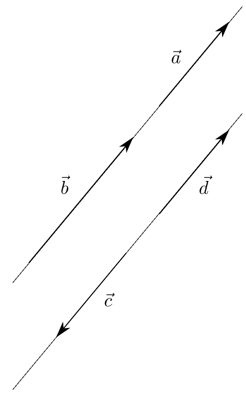
Notice that $\overrightarrow{BA} = -\overrightarrow{AB}$; this is true in general.
Here's a sketch of the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$:



$\overrightarrow{AB}$ and $\overrightarrow{CD}$ are both $(1,2)$; in the picture, you can see that the arrows which represent the vectors have the same length and the same direction. $\square$

Geometrically, two vectors (thought of as arrows) are **equal** if they have the same length and point in the same direction.

Example. In the picture below, assume the two lines are parallel. Which of the vectors $\vec{b}$, $\vec{c}$, $\vec{d}$ is equal to the vector $\vec{a}$?



$\vec{b}$ is not equal to $\vec{a}$; it has the same direction, but not the same length.
$\vec{c}$ is not equal to $\vec{a}$; it has the same length, but the opposite direction.
$\vec{d}$ is equal to $\vec{a}$, since it has the same length and direction. $\square$

Algebraically, two vectors are **equal** if their corresponding components are equal.

Example. Find a and b such that

$$(a + 2b, a - b) = (-8, 7).$$

Set the corresponding components equal and solve for a and b:

$$\begin{array}{cccccc} & a & + & 2b & = & -8 \\ - & a & - & b & = & 7 \\ \hline & & & 3b & = & -15 \\ / & & & 3 & & 3 \\ \hline & & & b & = & -5 \end{array}$$

Substituting this into $a + 2b = -8$, I get $a - 10 = -8$, so $a = 2$.
The solution is $a = 2$, $b = -5$. $\square$

The **length** of a geometric vector is the length of the arrow that represents it.
The **length** of an algebraic vector is given by the distance formula. If $\vec{v} = (a, b, c)$, the length of $\vec{v}$ is

$$\|\vec{v}\| = \sqrt{a^2 + b^2 + c^2}.$$

A vector with length 1 is called a **unit vector**.

Example. (a) Find the length of $(3, 12, -4)$.

(b) Show that $\left(\dfrac{-4}{5}, \dfrac{3}{5}\right)$ is a unit vector.

(a)

$$\|(3, 12, -4)\| = \sqrt{3^2 + 12^2 + (-4)^2} = 13. \quad \square$$

(b)

$$\left\|\left(\frac{-4}{5}, \frac{3}{5}\right)\right\| = \sqrt{\left(\frac{-4}{5}\right)^2 + \left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25} + \frac{9}{25}} = 1. \quad \square$$

Algebraically, you **add** or **subtract** vectors by adding or subtracting corresponding components:

$$(a, b) + (c, d) = (a + c, b + d), \quad (a, b) - (c, d) = (a - c, b - d).$$

(Use an analogous procedure to add or subtract 3-dimensional vectors.) You can't add or subtract vectors with different numbers of components. For example, you can't add a 2 dimensional vector to a 3 dimensional vector.

Algebraically, you **multiply a vector by a number** by multiplying each component by the number:

$$k \cdot (a, b) = (ka, kb).$$

Vectors that are multiples are said to be **parallel**.

Example. Compute:

(a) $(1, 5) + (7, 19)$.

(b) $(2, -3, 8) - (16, 11, 0)$.

(c) $6 \cdot (5, 3)$.

(d) $2 \cdot (2, -1, 2) + 4 \cdot (1, -1, 3)$.

(a)

$$(1, 5) + (7, 19) = (1 + 7, 5 + 19) = (8, 24).$$

(b)

$$(2, -3, 8) - (16, 11, 0) = (2 - 16, -3 - 11, 8 - 0) = (-14, -14, 8).$$

(c)

$$6 \cdot (5, 3) = (6 \cdot 5, 6 \cdot 3) = (30, 18).$$

(d)

$$2 \cdot (2, -1, 2) + 4 \cdot (1, -1, 3) = (4, -2, 4) + (4, -4, 12) = (8, -6, 16). \quad \square$$

Here are some properties of vector arithmetic. There is nothing surprising here.

Proposition. Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors (in the same space) and let k be a real number.

(a) (Associativity) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$.

(b) (Commutativity) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.

(c) (Zero vector) The vector $\vec{0}$ with all-0 components satisfies $\vec{0} + \vec{u} = \vec{u}$ and $\vec{u} + \vec{0} = \vec{u}$.

(d) (Additive inverse) The additive inverse $-\vec{u}$ of $\vec{u}$ is the vector whose components are the negatives of the components of $\vec{u}$. It satisfies $\vec{u} + (-\vec{u}) = \vec{0}$.

(e) (Distributivity) $k \cdot (\vec{u} + \vec{v}) = k \cdot \vec{u} + k \cdot \vec{v}$.

Note: To say that the vectors are *in the same space* means that, for example, $\vec{u}$, $\vec{v}$, and $\vec{w}$ are all vectors in $\mathbb{R}^3$. But all of the results are true if $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^{100}$ (100-dimensional Euclidean space).

Proof. The idea in all these cases is to write the vectors in component form and do the computation. For example, here is a proof of (c) in the case that $\vec{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$.

$$\vec{0} + \vec{u} = (0,0,0) + (u_1, u_2, u_3) = (u_1, u_2, u_3) = \vec{u}.$$

$$\vec{u} + \vec{0} = (u_1, u_2, u_3) + (0,0,0) = (u_1, u_2, u_3) = \vec{u}.$$

Here is a proof of (e). I'll consider the special case where $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^3$. Thus,

$$\vec{u} = (u_1, u_2, u_3) \quad \text{and} \quad \vec{v} = (v_1, v_2, v_3).$$

Then

$$\begin{aligned} k \cdot (\vec{u} + \vec{v}) &= k \cdot [(u_1, u_2, u_3) + (v_1, v_2, v_3)] \\ &= k \cdot (u_1 + v_1, u_2 + v_2, u_3 + v_3) \\ &= (k \cdot (u_1 + v_1), k \cdot (u_2 + v_2), k \cdot (u_3 + v_3)) \\ &= (k \cdot u_1 + k \cdot v_1, k \cdot u_2 + k \cdot v_2, k \cdot u_3 + k \cdot v_3) \\ &= (k \cdot u_1, k \cdot u_2, k \cdot u_3) + (k \cdot v_1, k \cdot v_2, k \cdot v_3) \\ &= k \cdot (u_1, u_2, u_3) + k \cdot (v_1, v_2, v_3) \\ &= k \cdot \vec{u} + k \cdot \vec{v} \end{aligned}$$

The other parts are proved in similar fashion. □

There is an alternate notation for vectors that is often used in physics and engineering. $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are the **unit vectors** in the x, y, and z directions:

$$\hat{\imath} = (1,0,0), \quad \hat{\jmath} = (0,1,0), \quad \hat{k} = (0,0,1).$$

Note that

$$(a,b,c) = a \cdot (1,0,0) + b \cdot (0,1,0) + c \cdot (0,0,1) = a\hat{\imath} + b\hat{\jmath} + c\hat{k}.$$

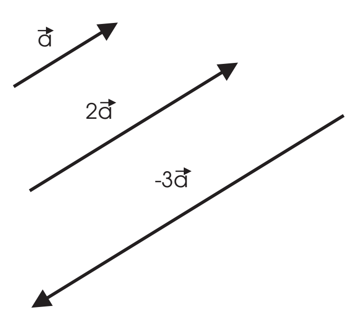
For example,

$$(-3,6,10) = -3\hat{\imath} + 6\hat{\jmath} + 10\hat{k}.$$

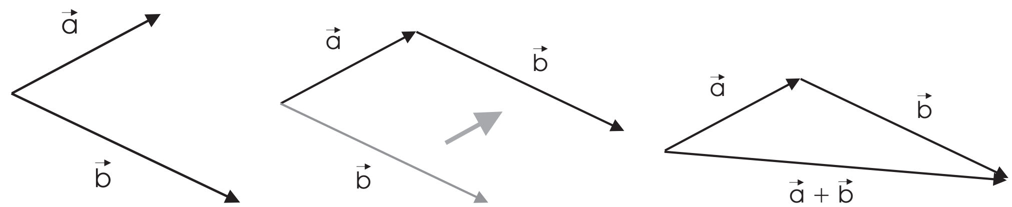
In 2 dimensions, $(a,b) = a\hat{\imath} + b\hat{\jmath}$. There is no $\hat{\imath}$-$\hat{\jmath}$-$\hat{k}$ notation for vectors with more than 3 components. You operate with vectors using the $\hat{\imath}$-$\hat{\jmath}$-$\hat{k}$ notation in the obvious ways. For example,

$$2(3\hat{\imath} - 4\hat{k}) + (5\hat{\imath} + 7\hat{\jmath} - 11\hat{k}) = 11\hat{\imath} + 7\hat{\jmath} - 19\hat{k}.$$

Geometrically, **multiplying a vector by a number** multiplies the length of the arrow by the number. In addition, if the number is negative, the arrow's direction is reversed:

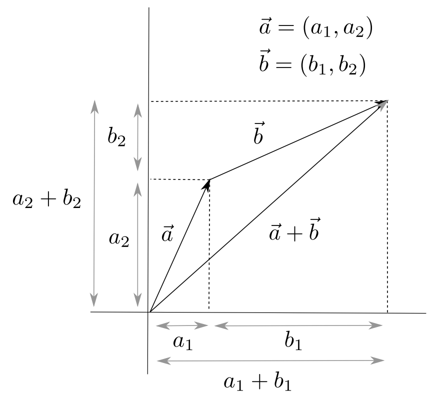


You **add** geometric vectors as shown below. Move one of the vectors — say $\vec{b}$ — keeping its length and direction unchanged so that it starts at the end of the other vector. Since the copy has the same length and direction as the original $\vec{b}$, it's equal to $\vec{b}$.

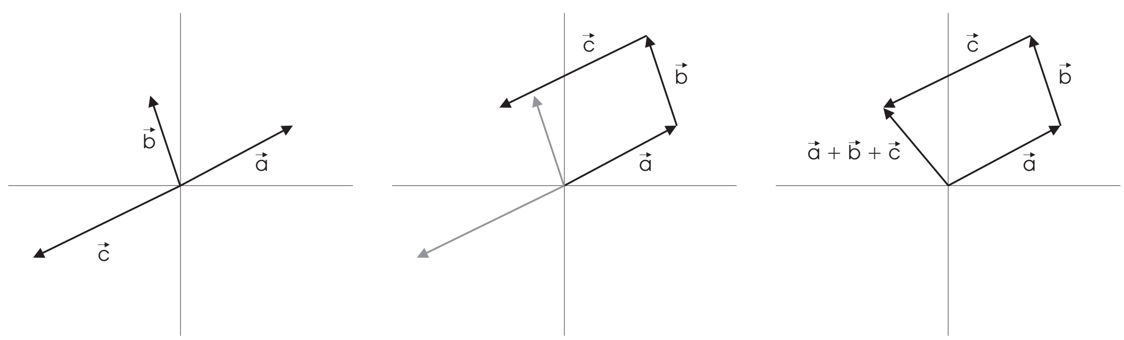


Next, draw the vector which starts at the starting point of $\vec{a}$ and ends at the tip of $\vec{b}$. This vector is the sum $\vec{a} + \vec{b}$.

The picture below illustrates why the geometric addition rule follows from the algebraic addition rule. It is obviously a special case with two 2-dimensional vectors with positive components, but I think it makes the result plausible.

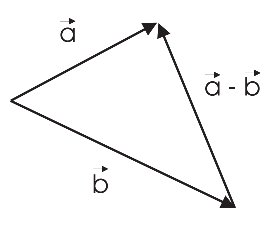


To add several vectors, move the vectors (keeping their lengths and directions unchanged) so that they are "head-to-tail". In the second picture below, I moved $\vec{b}$ and $\vec{c}$.



Finally, draw a vector from the start of the first vector to the end of the last vector. That vector is the sum — in this case, $\vec{a} + \vec{b} + \vec{c}$.

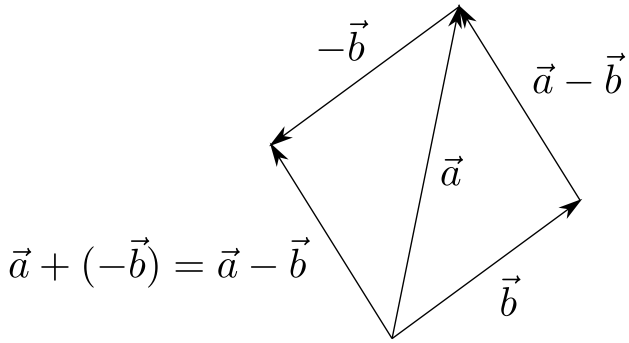
The picture below shows how to subtract one vector from another — in this case, $\vec{a} - \vec{b}$ is the vector which goes from the tip of $\vec{b}$ to the tip of $\vec{a}$.



There are a couple of ways to see this. First, if you interpret this as an addition picture using the "head-to-tail" rule, it says

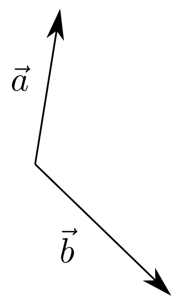
$$\vec{b} + (\vec{a} - \vec{b}) = \vec{a}.$$

Alternatively, construct $-\vec{b}$ by "flipping" $\vec{b}$ around, then add $-\vec{b}$ to $\vec{a}$.

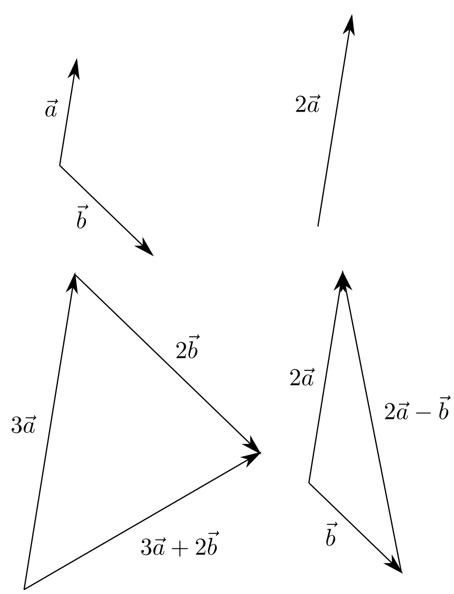


This gives $\vec{a} + (-\vec{b}) = \vec{a} - \vec{b}$. As the picture shows, it is the same as the vector from the head of $\vec{b}$ to the head of $\vec{a}$, because the two vectors are opposite sides of a parallelogram.

Example. Vectors $\vec{a}$ and $\vec{b}$ are shown in the picture below.



Draw pictures of the vectors $2\vec{a}$, $3\vec{a} + 2\vec{b}$, and $2\vec{a} - \vec{b}$.



□