Velocity and Acceleration

A function $f : \mathbb{R} \to \mathbb{R}^n$ can be thought of as a curve in $\mathbb{R}^n$. Write the curve in parametric form

$$f(t) = (f_1(t), f_2(t), \ldots f_n(t)).$$

Think of the parameter t as *time* and the curve as being traced out by a moving object, so that the object is at the position $f(t)$ at time t.

With this interpretation:

(a) $f'(t)$ is the **velocity vector** of the object. It points in the direction that the object is travelling at time t. Its length is the **speed** of the object at time t.

(b) $f''(t)$ is the **acceleration vector** of the object. It represents the direction and magnitude of the change of the velocity vector at time t.

Example. Find the velocity and acceleration vectors at $t = 3$ for the curve with position function

$$f(t) = \left(e^{2t} + t, \frac{1}{t+1}, (2t-5)^2\right).$$

The velocity is the derivative of the position:

$$v(t) = f'(t) = \left(2e^{2t} + 1, -\frac{1}{(t+1)^2}, 4(2t-5)\right).$$

The acceleration is the derivative of the velocity:

$$a(t) = f''(t) = \left(4e^{2t}, \frac{2}{(t+1)^3}, 8\right).$$

Setting $t = 3$ gives

$$v(3) = \left(2e^6 + 1, -\frac{1}{16}, 4\right) \quad \text{and} \quad a(3) = \left(4e^6, \frac{1}{32}, 8\right). \quad \square$$

Example. The position of an evil lime jello at time t is

$$f(t) = (2t^3 - 3t^2 + 5, 7t + 1, t^2 + 3t).$$

Find its velocity vector and its speed at $t = 2$.

The velocity is the derivative of the position:

$$v(t) = f'(t) = (6t^2 - 6t, 7, 2t + 3).$$

The velocity at $t = 2$ is

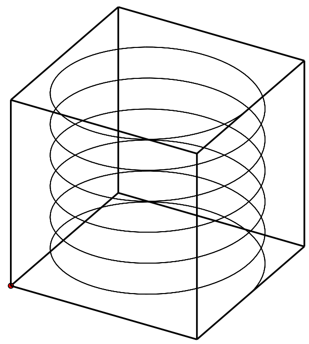
$$v(2) = f'(2) = (12, 7, 7).$$

The speed is the length of the velocity vector:

$$\|v(2)\| = \sqrt{12^2 + 7^2 + 7^2} = \sqrt{242}. \quad \square$$

Example. A cheeseburger moves on a **circular helix** given by

$$f(t) = (\cos 2t, \sin 2t, 3t).$$



Show that it moves with constant speed.

The velocity vector is

$$v(t) = f'(t) = (-2\sin 2t, 2\cos 2t, 3).$$

The speed is the length of the velocity vector, which is

$$\|v(t)\| = \sqrt{4(\sin 2t)^2 + 4(\cos 2t)^2 + 9} = \sqrt{13}.$$

Thus, the cheeseburger moves with constant speed. □

Example. Prove that if a curve $f(t)$ has constant length, then its velocity and position vectors are always perpendicular.

I'll use the fact that the square of the length of a vector equals the dot product of the vector with itself:

$$\|v\|^2 = v \cdot v.$$

Suppose $\|f(t)\| = r$, where r is a constant. Then use the identity above, differentiate, and apply the Product Rule for dot products:

$$\begin{aligned} \|f(t)\|^2 &= r^2 \\ f(t) \cdot f(t) &= r^2 \\ \frac{d}{dt}(f(t) \cdot f(t)) &= \frac{d}{dt} r^2 \\ f'(t) \cdot f(t) + f(t) \cdot f'(t) &= 0 \\ 2f(t) \cdot f'(t) &= 0 \\ f(t) \cdot f'(t) &= 0 \end{aligned}$$

Since $f(t)$ and $f'(t)$ have dot product 0, they are perpendicular. □

Note: To say that $f(t)$ has constant length r means that a point on the curve stays a constant distance r from the origin. Thus, it must be moving on the sphere of radius r centered at the origin.

Since $v(t) = f'(t)$, you can integrate to find the position function from the velocity:

$$f(t) = \int v(t)\, dt.$$

Likewise, since $a(t) = v'(t)$, you can integrate to find the velocity from the acceleration:

$$v(t) = \int a(t)\,dt.$$

Antiderivatives are only determined up to an arbitrary constant. But you may be able to determine the arbitrary constant if you are given **initial conditions**.

Example. The acceleration vector for a bacon quiche is

$$a(t) = (30t, 6)\,.$$

Find the position function $f(t)$, if $v(1) = (21, 7)$ and $f(1) = (9, 8)$.

$$v(t) = \int a(t)\,dt = \int (30t, 6)\ dt = \left(15t^2, 6t\right) + (c_1, c_2).$$

Now $v(1) = (21, 7)$, so

$$\begin{aligned} (21, 7) &= (15 \cdot 1^2, 6 \cdot 1) + (c_1, c_2) \\ (21, 7) &= (15, 6) + (c_1, c_2) \\ (6, 1) &= (c_1, c_2) \end{aligned}$$

Hence,

$$v(t) = \left(15t^2, 6t\right) + (6, 1) = \left(15t^2 + 6, 6t + 1\right).$$

Next,

$$f(t) = \int v(t)\,dt = \int \left(15t^2 + 6, 6t + 1\right)\ dt = \left(5t^3 + 6t, 3t^2 + t\right) + (d_1, d_2).$$

Now $f(1) = (9, 8)$, so

$$\begin{aligned} (9, 8) &= \left(5 \cdot 1^3 + 6 \cdot 1, 3 \cdot 1^2 + 1\right) + (d_1, d_2) \\ (9, 8) &= (11, 4) + (d_1, d_2) \\ (-2, 4) &= (d_1, d_2) \end{aligned}$$

Hence,

$$f(t) = \left(5t^3 + 6t, 3t^2 + t\right) + (-2, 4) = \left(5t^3 + 6t - 2, 3t^2 + t + 4\right). \quad \square$$
