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Complex Numbers

In this section, I'll review basic properties of the complex numbers. The emphasis will be on the computational tools needed to use complex numbers in linear algebra.

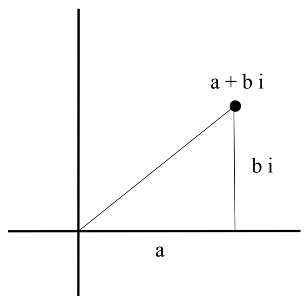
A **complex number** is a number of the form $a + bi$, where a and b are real numbers. a is called the **real part** and b is called the **imaginary part**; the notation is

$$\text{re}(a + bi) = a, \quad \text{im}(a + bi) = b.$$

i is a special symbol, sometimes called the complex or imaginary unit. It behaves somewhat like a variable, but it has a special multiplication property we'll see below.

The word "imaginary" is traditional, but not very good: It might lead you to believe that there's something "fake" about complex numbers, as opposed to the *real* numbers. In fact, the complex numbers are a number system like the real numbers or the integers, and there is nothing "fake" about them: I'll sketch a construction of the complex numbers using matrices below.

You can picture a complex number $a+bi$ as a point in the x-y-plane; we think of "a" as the x-coordinate and bi (or b) as the y-coordinate.



Thus, the y-axis is an "i-axis".

The standard notation for the complex numbers is $\mathbb{C}$.

You add, subtract, and multiply complex numbers in the following ways:

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i,$$

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

In the last equation, take $a = c = 0$ and $b = d = 1$. The equation becomes

$$i \cdot i = -1.$$

This means that $i^2 = -1$, so i is described as "the square root of -1". You might object that -1 doesn't have a square root. What is true is that *no real number* is a square root of -1, but the complex numbers are a different number system.

You can divide by (nonzero) complex numbers by "multiplying the top and bottom by the conjugate":

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{(ac + bd) + (-ad + bc)i}{c^2 + d^2}.$$

With these operations, the set of complex numbers forms a **field**.

At this point, you might be a bit suspicious — how can we just make up a bunch of symbols and operations and call the result "the complex numbers"? Here's a sketch which describes how you can "build" the complex numbers using matrices. The complex number $a + bi$ will correspond to the real 2×2 matrix

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

Setting $a = 0$ and $b = 1$, we see that the complex number i should correspond to the matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The operations will be matrix addition and multiplication. Let's try them out:

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ -b-d & a+c \end{bmatrix},$$

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \begin{bmatrix} ac-bd & ad+bc \\ -ad-bc & ac-bd \end{bmatrix}.$$

Notice that we're getting the same expressions we gave above for addition and multiplication of complex numbers. Thus, if you're comfortable with matrices and matrix arithmetic, you can think of "$a+bi$" notation as shorthand for the matrix equivalents above. In an abstract algebra course, you'll probably see another construction of the complex numbers using **quotient rings**. In linear algebra, we'll just use the "$a+bi$" form of complex numbers, as it is the simplest for computations.

To continue with our discussion of complex number arithmetic, I'll note here that the **conjugate** of complex number is obtained by flipping the sign of the imaginary part. The conjugate of $a + bi$ is denoted $\overline{a + bi}$ or sometimes $(a + bi)^*$. Thus,

$$\overline{a + bi} = a - bi.$$

The **norm** of a complex number is

$$|a + bi| = \sqrt{a^2 + b^2}.$$

Note that

$$(a + bi)(\overline{a + bi}) = (a + bi)(a - bi) = a^2 + b^2 = |a + bi|^2.$$

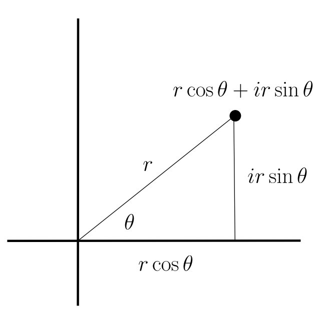
When a complex number is written in the form $a + bi$, it's said to be in **rectangular form**. There is another form for complex numbers that is useful: The polar form $re^{i\theta}$. In this form, r and θ have the same meanings that they do in polar coordinates.

DeMoivre's formula relates the polar and rectangular forms:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

This key result can be proven, for example, by expanding both sides in power series. Using this, we get

$$re^{i\theta} = r\cos\theta + i \cdot r\sin\theta.$$



This relates the rectangular and polar forms of a complex number.

Note also that

$$|e^{i\theta}| = |\cos\theta + i\sin\theta| = \sqrt{(\cos\theta)^2 + (\sin\theta)^2} = 1.$$

Thus, $e^{i\theta}$ is a complex number of norm 1.

Example. Convert $3+4i$ to polar form.

$$3+4i = |3+4i| \cdot \frac{3+4i}{|3+4i|} = 5\left(\frac{3}{5}+\frac{4}{5}i\right).$$

Let $\theta = \sin^{-1}\dfrac{4}{5}$ (or $\cos^{-1}\dfrac{3}{5}$). Then

$$3+4i = 5(\cos\theta + i\sin\theta). \quad \square$$

In the examples that follow, we'll the polar and complex exponential forms of complex numbers to simplify algebraic computations, derive trig identities, and compute integrals.

Example. (**A trick with Demoivre's formula**) Find $(1+i\sqrt{3})^8$.

It would be tedious to try to multiply this out. Instead, I'll try to write the expression in terms of $\cos\theta + i\sin\theta$ for a good choice of θ.

$$(1+i\sqrt{3})^8 = 2^8 \cdot \left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^8 = 256\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^8 = 256(e^{\pi i/3})^8 = 256e^{8\pi i/3} =$$

$$256\left(\cos\frac{8\pi}{3}+i\sin\frac{8\pi}{3}\right) = 256\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right) = 256\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) = -128+128i\sqrt{3}. \quad \square$$

Example. (**Proving trig identities**) Prove the **angle addition formulas**:

$$\cos(a+b) = \cos a\cos b - \sin a\sin b.$$

$$\sin(a+b) = \sin a\cos b + \sin b\cos a.$$

I have

$$\begin{aligned} e^{(a+b)i} &= e^{ai}e^{bi} \\ \cos(a+b)+i\sin(a+b) &= (\cos a + i\sin a)(\cos b + i\sin b) \\ \cos(a+b)+i\sin(a+b) &= (\cos a\cos b - \sin a\sin b) + i(\sin a\cos b + \sin b\cos a) \end{aligned}$$

Equating real and imaginary parts on the left and right sides, I get

$$\cos(a+b) = \cos a\cos b - \sin a\sin b \quad \text{(real parts).}$$

$$\sin(a+b) = \sin a\cos b + \sin b\cos a \quad \text{(imaginary parts).} \quad \square$$

Example. (**Computing integrals**) Compute $\displaystyle\int e^{2x}\cos 3x\,dx$.

Note that $\cos 3x = \operatorname{re}(e^{3xi})$. Thus,

$$\int e^{2x}\cos 3x\,dx = \int e^{2x}\operatorname{re}(e^{3xi})\,dx = \operatorname{re}\int e^{2x}e^{3xi}\,dx = \operatorname{re}\int e^{(2+3i)x}\,dx = \operatorname{re}\frac{1}{2+3i}e^{(2+3i)x}+c =$$

$$\operatorname{re}\frac{2-3i}{13}e^{2x}(\cos 3x + i\sin 3x) + c = \frac{1}{13}e^{2x}\operatorname{re}(2-3i)(\cos 3x + i\sin 3x) + c =$$

$$\frac{1}{13}e^{2x}\operatorname{re}\left((2\cos 3x + 3\sin 3x) + i(2\sin 3x - 3\cos 3x)\right) + c = \frac{1}{13}e^{2x}(2\cos 3x + 3\sin 3x) + c. \quad \square$$

Suppose we have a polynomial of degree n with real coefficients:

$$a^n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_i \in \mathbb{R}.$$

The **Fundamental Theorem of Algebra** says that the polynomial has n roots which are real or complex numbers, some of which may be repeated. For example, the roots of $x^3 - 8$ are

$$2, \quad -1+\sqrt{3}, \quad -1-\sqrt{3}.$$

In general, *finding* the roots may require numerical approximation. You may see a proof of this theorem in advanced algebra or analysis courses; we'll take the result for granted.