Cartesian Products

Definition. Let S and T be sets. The **Cartesian product** of S and T is the set $S \times T$ consisting of all ordered pairs (s, t), where $s \in S$ and $t \in T$.

Ordered pairs are characterized by the following property: $(a, b) = (c, d)$ if and only if

$$a = c \quad \text{and} \quad b = d.$$

Remarks. (a) $S \times T$ is not the same as $T \times S$ unless $S = T$.

(b) You can define an ordered pair using sets. For example, the ordered pair (x, y) can be defined as the set $\{x, \{x, y\}\}$.

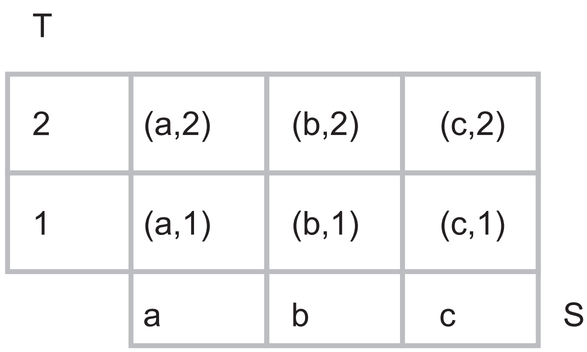
Example. Let $S = \{a, b, c\}$ and $T = \{1, 2\}$. List the elements of $S \times T$ and sketch the set.

$$S \times T = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}.$$

Notice that S and T are *not* subsets of $S \times T$. There are subset which "look like" S and T; for example, here's a subset that "looks like" S:

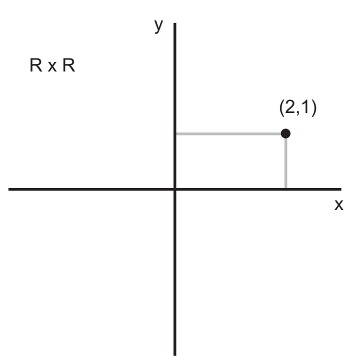
$$U = \{(a, 1), (b, 1), (c, 1)\}.$$

But this is not S: The elements of S are a, b, and c, whereas the elements of the subset U are *pairs*. Here's a picture of $S \times T$. The elements are points in the grid:



$\square$

$\mathbb{R} \times \mathbb{R}$ consists of all pairs (x, y), where $x, y \in \mathbb{R}$. This is the same thing as the the x-y-plane:



$\square$

Example. Consider the following subset of $\mathbb{R} \times \mathbb{R}$:

$$S = \{(2x, 5x) \mid x \in \mathbb{R}\}.$$

(a) Prove that $(-14, -35) \in S$.

(b) Prove that $(18, 50) \notin S$.

(a)

$$(-14, -35) = (2 \cdot (-7), 5 \cdot (-7)) \in S. \quad \square$$

(b) Suppose $(18, 50) \in S$. Then for some $x \in \mathbb{R}$, I have

$$(18, 50) = (2x, 5x).$$

Equating the first components, I get $2x = 18$, so $x = 9$. But equating the second components, I get $5x = 50$, so $x = 10$. This is a contradiction, so $(18, 50) \notin S$. $\square$

Example. $\mathbb{Z} \times \mathbb{Z}$ is the set of pairs (m, n) of integers. Consider the following subsets of $\mathbb{Z} \times \mathbb{Z}$:

$$A = \{(3n + 5, 9n + 10) \mid n \in \mathbb{Z}\} \quad \text{and} \quad B = \{(s, t) \in \mathbb{Z} \times \mathbb{Z} \mid s + t \quad \text{is odd}\}.$$

Prove that $A \subset B$.

Let $(3n + 5, 9n + 10) \in A$. B consists of pairs whose components add to an odd number. So I add the components of $(3n + 5, 9n + 10)$:

$$(3n + 5) + (9n + 10) = 12n + 15 = 2(6n + 7) + 1.$$

Since $2(6n + 7) + 1$ is odd, $(3n + 5) + (9n + 10)$ is odd. This proves that $(3n + 5, 9n + 10) \in B$. $\square$

You can take the product of more than 2 sets — even an infinite number of sets, though I won't consider infinite products here.

For example, $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ consists of **ordered triples** (a, b, c), where a, b, and c are integers.

Example. Consider the following subset of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$:

$$W = \{(a, b, a + 2b) \mid a, b \in \mathbb{Z}\}.$$

(a) Show that $(-3, 5, 7) \in W$.

(b) Show that $(2, -4, 6) \notin W$.

(a)

$$(-3, 5, 7) = (-3, 5, -3 + 2 \cdot 5) \in W. \quad \square$$

(b) Suppose $(2, -4, 6) \in W$. Then for some integers a and b, I have

$$(2, -4, 6) = (a, b, a + 2b).$$

Equating components, I get three equations:

$$a = 2, \quad b = -4, \quad a + 2b = 6.$$

But substituting $a = 2$ and $b = -4$ into $a + 2b$ gives

$$a + 2b = 2 + 2 \cdot (-4) = -6 \neq 6.$$

This contradiction proves that $(2, -4, 6) \notin W$. $\square$
