Counterexamples

A **counterexample** is an example that disproves a universal ("for all") statement. Obtaining counterexamples is a very important part of mathematics, because doing mathematics requires that you develop a *critical attitude* toward claims. When you have an idea or when someone tells you something, *test the idea* by trying examples. If you find a counterexample which shows that the idea is false, *that's good*: Progress comes not only through doing the right thing, but also by correcting your mistakes.

Suppose you have a quantified statement:

"All x's satisfy property P": $\forall x P(x)$.

What is its negation?

$\sim \forall x P(x) \leftrightarrow \exists x \sim P(x)$.

In words, the second quantified statement says: "There is an x which does not satisfy property P". In other words, to prove that "All x's satisfy property P" is *false*, you must find an x which *does not* satisfy property P.

Example. Explain what you must do to disprove the statement:

(a) "All professors like pizza".

(b) "For every real number x, $(x+1)^2 = x^2 + 1$".

(c) "$x^3 + 5x + 3$ has a root between $x = 0$ and $x = 1$".

(a) To disprove "All professors like pizza", you must find a professor who does not like pizza. □

(b) To disprove the statement "For every real number x, $(x+1)^2 = x^2 + 1$", you must find a real number x for which $(x+1)^2 \neq x^2 + 1$. □

(c) To disprove the statement "$x^3 + 5x + 3$ has a root between $x = 0$ and $x = 1$", it's not enough to say "$x = 0.5$ is between $x = 0$ and $x = 1$, but $(0.5)^3 + 5(0.5) + 3 = 5.625 \neq 0$". The statement to be disproved is an *existence* statement:

"There is an x such that $0 < x < 1$ and $x^3 + 5x + 3 = 0$."

You can check that the negation is:

"For all x, it is not the case that both $0 < x < 1$ and $x^3 + 5x + 3 = 0$."

To *disprove* the original statement is to *prove* its negation, but a single example will not prove this "for all" statement. □

The point made in the last example illustrates the difference between "proof by example" — which is usually invalid — and giving a counterexample.

(a) A single example can't *prove* a universal statement (unless the universe consists of only one case!).

(b) A single counterexample can *disprove* a universal statement.

In many cases where you need a counterexample, the statement under consideration is an if-then statement.So how do you give a counterexample to a conditional statement $P \to Q$? By basic logic, $P \to Q$ is false when P is true and Q is false. Therefore:

To give a counterexample to a conditional statement $P \to Q$, find a case where P is true but Q is false.

Equivalently, here's the rule for negating a conditional:

$$\sim (P \to Q) \leftrightarrow (P \wedge \sim Q)$$

Again, you need the "if-part" P to be true and the "then-part" Q to be false (that is, $\sim Q$ must be true).

Example. Give a counterexample to the statement

"If n is an integer and n^2 is divisible by 4, then n is divisible by 4."

To give a counterexample, I have to find an integer n such n^2 is divisible by 4, but n is *not* divisible by 4 — the "if" part must be true, but the "then" part must be false. Consider $n = 6$. Then $n^2 = 36$ is divisible by 4, but $n = 6$ is not divisible by 4. Thus, $n = 6$ is a counterexample to the statement.

On the other hand, consider $n = 5$. While $n = 5$ is not divisible by 4, $n^2 = 25$ is also not divisible by 4. For $n = 5$, the "if" and "then" parts of the statement are both false. Therefore, $n = 5$ is not a counterexample to the statement. □

Example. Consider real-valued functions defined on the interval $0 \le x \le 1$. Give a counterexample to the following statement:

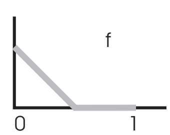
"If the product of two functions is the zero function, then one of the functions is the zero function."

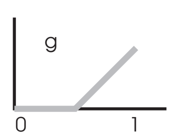
(The *zero function* is the function which produces 0 for all inputs — i.e. the constant function $f = 0$.)

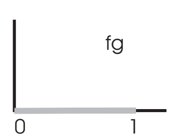
Here are two functions whose product is the zero function, neither of which is the zero function:

$$f(x) = \begin{cases} \dfrac{1}{2} - x & \text{if } 0 \le x \le \dfrac{1}{2} \\ 0 & \text{if } \dfrac{1}{2} < x \le 1 \end{cases} \qquad g(x) = \begin{cases} 0 & \text{if } 0 \le x \le \dfrac{1}{2} \\ x - \dfrac{1}{2} & \text{if } \dfrac{1}{2} < x \le 1 \end{cases}.$$

Here's a picture which makes it clear why their product is always 0:







□

Example. Give a counterexample to the following statement:

"If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges."

You may recall this mistake from studying infinite series.

The harmonic series is

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots.$$

It diverges, even though $\lim_{n\to\infty} \frac{1}{n} = 0$. □

The *converse* of the given statement — the Zero Limit Test — *is* true: If $\sum_{n=1}^{\infty} \frac{1}{n}$ converges, then $\lim_{n\to\infty} a_n = 0$. Or to put it another way (taking the contrapositive), if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges*.

For example, the series $\sum_{n=1}^{\infty} \frac{3n^2+1}{4n^2-3}$ diverges, because

$$\lim_{n\to\infty} \frac{3n^2+1}{4n^2-3} = \frac{3}{4} \neq 0. \quad □$$

An **algebraic identity** is an equation which is true for all values of the variables for which both sides of the equation are defined.

For example, here is an algebraic identity for real numbers:

$$\frac{1}{x} + 1 = \frac{x+1}{x}.$$

It is true for all $x \neq 0$.

Since an algebraic identity is a statement about *all* numbers in a certain set, you can prove that a statement is *not* an identity by producing a counterexample.

Example. Prove that "$(a+b)^2 = a^2 + b^2$" is *not* an algebraic identity, where $a, b \in \mathbb{R}$.

I need to find *specific* real numbers a and b for which the equation is false.

If an equation is *not* an identity, you can usually find a counterexample by trial and error. In this case, if $a = 1$ and $b = 2$, then

$$(a+b)^2 = (1+2)^2 = 3^2 = 9, \quad \text{while} \quad a^2 + b^2 = 1^2 + 2^2 = 5.$$

So if $a = 1$ and $b = 2$, then $(a+b)^2 \neq a^2 + b^2$, and hence the statement is not an identity.

A common mistake is to say:

"$(a+b)^2 = a^2 + 2ab + b^2$, which is not the same as $a^2 + b^2$."

In the first place, how do you *know* $a^2 + 2ab + b^2$ is not the same as $a^2 + b^2$? It is no answer to say that they *look* different — after all, $(\sin\theta)^2 + (\cos\theta)^2$ looks very different than 1, but $(\sin\theta)^2 + (\cos\theta)^2 = 1$ *is* an identity.

In the second place, $a^2 + 2ab + b^2$ *is* the same as $a^2 + b^2$ if (for instance) $a = 17$ and $b = 0$ — and they're equal for many other values of a and b.

To disprove an identity, you should always give a *specific numerical counterexample.* □

Example. Give a counterexample which shows that "$\dfrac{1}{x+2} = \dfrac{1}{x} + \dfrac{1}{2}$" is not an identity.

An identity is only asserted for values of the variables for which both sides are defined. So the assertion here is actually

"$\dfrac{1}{x+2} = \dfrac{1}{x} + \dfrac{1}{2}$ for $x \ne 0$ and $x \ne -2$."

Thus, $x = 1$ is a counterexample, since

$$\frac{1}{x+2} = \frac{1}{3}, \quad \text{while} \quad \frac{1}{x} + \frac{1}{2} = \frac{1}{1} + \frac{1}{2} = \frac{3}{2}.$$

You should *not* give $x = 0$ or $x = -2$ as a counterexample. For these values of x, one side of the purported identity is undefined. Therefore, these cases are not part of what is claimed, so they can't be counterexamples. □

Finally, do not confuse giving a counterexample with *proof by contradiction.* A counterexample *disproves* a statement by giving a situation where the statement is false; in proof by contradiction, you *prove* a statement by assuming its negation and obtaining a contradiction.