2-12-2020

Existence Proofs

An **existence proof** shows that an object exists. In some cases, this means displaying the object, or giving a method for finding it.

Example. Show that there is a real number x such that $\sin x > 0$ but $\cos x < 0$.

There are many possibilities; for example,

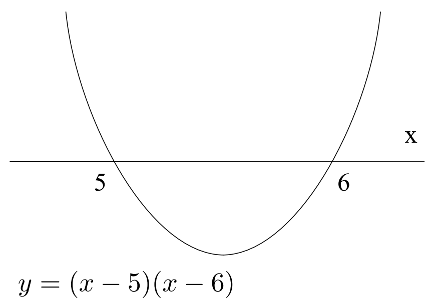
$$\sin \frac{3\pi}{4} = \frac{\sqrt{2}}{2}, \quad \text{but} \quad \cos \frac{3\pi}{4} = -\frac{\sqrt{2}}{2}. \quad \square$$

Example. Show that there is a real number x such that $x^2 + 30 < 11x$.

Rewrite the inequality as

$$x^2 - 11x + 30 < 0, \quad \text{or} \quad (x-5)(x-6) < 0.$$

The graph of $y = (x-5)(x-6)$ looks like this:



The graph lies below the x-axis between $x = 5$ and $x = 6$. So, for example, $x = 5.5$ meets the conditions:

$$5.5^2 - 11 \cdot 5.5 + 30 = -0.25 < 0. \quad \square$$

In some cases, you can know that an object exists without having any way of finding it (or finding it exactly). By analogy:

(a) If you throw your keys into a corn field, you *know* your keys are in the field — but you may have trouble finding them!

(b) You know that Calvin Butterball has a birthday, even though you don't know *what day* it is.

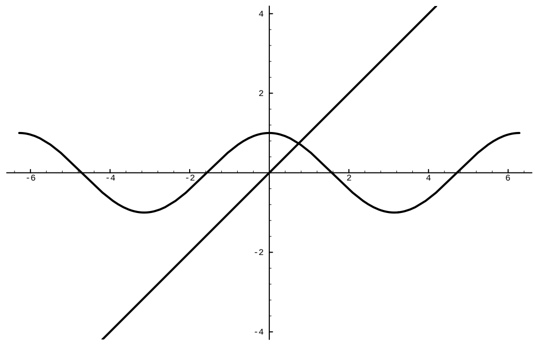
You've seen results of this kind in calculus. One such result is:

Theorem. (**The Intermediate Value Theorem:**) Let f be a continuous function on the interval $[a, b]$. Suppose that c is a number between $f(a)$ and $f(b)$. Then $f(x) = c$ for some x in the interval $[a, b]$.

The Intermediate Value Theorem does not tell you how to *find* an x such that $f(x) = c$ — it simply *guarantees* that such an x exists.

Example. Show that there is a real number x such that $x = \cos x$.

The assertion means that the graphs of $y = x$ and $y = \cos x$ intersect:



It looks like they do. Note, however, that a picture is not a proof.

Let $f(x) = \cos x - x$. Then

$$f(0) = \cos 0 - 0 = 1, \quad \text{while} \quad f(\pi) = \cos \pi - \pi = -1 - \pi.$$

Since $f(0)$ is positive and $f(\pi)$ is negative, and since f is continuous for all x, the Intermediate Value Theorem implies that there is an x between 0 and π for which $f(x) = 0$. Then $\cos x - x = 0$, so $\cos x = x$.

Notice that the Intermediate Value Theorem doesn't tell you what x is, or how to find it. (It's approximately 0.73909.) $\square$

Example. Suppose f is a continuous function satisfying

$$f(5) = 11 \quad \text{and} \quad f(8) = -20.$$

Prove that there is a number c such that $5 \le c \le 8$ and

$$3c + f(c) = 10.$$

The function $g(x) = 3x + f(x)$ is continuous.

$$g(5) = 3 \cdot 5 + f(5) = 15 + 11 = 26 \quad \text{and} \quad g(8) = 3 \cdot 8 + (-20) = 4.$$

Since 10 is between 26 and 4, there is a number c such that $5 \le c \le 8$ and

$$g(c) = 3c + f(c) = 10. \quad \square$$

To say that there is an x satisfying a certain property does not mean that there is *only one* x satisfying the property. If that is what is meant, it has to be stated explicitly. Hence, there might be *many* values which satisfy the conclusion of the Intermediate Value Theorem.

Here's another existence theorem from calculus:

Theorem. (**Mean Value Theorem**) Suppose f is function which is continuous on the closed interval $a \le x \le b$ and differentiable on the open interval $a < x < b$. Then there is a number c such that $a < c < b$ and

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Example. Find a number c which satisfies the conclusion of the Mean Value Theorem when it is applied to $f(x) = x^3$ on the interval $1 \le x \le 3$.

Note that

$$\frac{f(3) - f(1)}{3 - 1} = 13.$$

Now $f'(x) = 3x^2$, so setting $3c^2 = 13$, I find that $c = \pm\sqrt{\frac{13}{3}}$. Both of these values satisfy the conclusion of the Mean Value Theorem. $\square$

Example. Suppose f is a differentiable function satisfying

$$f(-2) = 10 \quad \text{and} \quad f'(x) > 3 \quad \text{for all} \quad x.$$

Prove that $f(5) > 31$.

Applying the Mean Value Theorem to f on the interval $-2 \le x \le 5$ gives a number c such that $-2 < c < 5$ and

$$\frac{f(5) - f(-2)}{5 - (-2)} = f'(c).$$

Then

$$\begin{aligned} \frac{f(5) - f(-2)}{5 - (-2)} &= f'(c) > 3 \\ \frac{f(5) - 10}{7} &> 3 \qquad \square \\ f(5) - 10 &> 21 \\ f(5) &> 31 \end{aligned}$$

Rolle's theorem is special case of the Mean Value Theorem: With the assumptions of the theorem, if $f(a) = f(b)$, then there is a number c such that $a < c < b$ and

$$f'(c) = 0.$$

That is, c is a **critical point** of f.

Example. Let $f(x) = x^2 \sin x$. Prove that there is a number c between 1 and π such that $f'(c) = 0$.

f is differentiable. Moreover,

$$f(0) = 0 \quad \text{and} \quad f(\pi) = 0.$$

By Rolle's theorem, there is a number c between 0 and π such that $f'(c) = 0$. $\square$

In the last example, I *found* numbers satisfying the conclusion of the theorem — but again, there is no guarantee that I can find such numbers explicitly. The theorem just says that at least one such number *exists*.