3-20-2020

Infinite Unions and Intersections

The set constructions I've considered so far — things like $A \cup B$, $\overline{C}$, $D \cap E$ — have involved finite numbers of sets. It's often necessary to work with infinite collections of sets, and to do this, you need a way of naming them and keeping track of them.

Definition. Let I be a set. A **collection of sets indexed by** I consists of a collection of sets S_i, one set S_i for each element $i \in I$.

You could make this more precise by defining a collection of sets indexed by I to be a function from I to the class of all sets. I'll stick with this informal definition, since it won't cause us any difficulties in what we do.

Let $I = \{1, 2, 3, 4\}$. A collection of sets indexed by I consists of four sets S_1, S_2, S_3, and S_4. For example,

$$S_1 = \{1, 2, 3\}, \quad S_2 = \{a, b, c\}, \quad S_3 = \mathbb{R}, \quad S_4 = \{1, 2, 3\}.$$

Note that $S_1 = S_3$; some of the sets in the collection may be identical.

Here's another collection of sets indexed by I:

$$S_1 = \emptyset, \quad S_2 = \mathbb{Z}, \quad S_3 = \{\pi, e\}, \quad S_4 = \{\text{pepperoni}, \text{sausage}\}.$$

This would not be very interesting if I were only considering finite collections of sets. Here are some infinite collections of sets.

Let $I = \mathbb{N} = \{1, 2, 3, \ldots\}$. A collection of sets indexed by I is an infinite collection of sets S_1, S_2, S_3, S_4, ...

Here is a collection of sets indexed by $\mathbb{N}$:

$$S_1 = (0, 1), \quad S_2 = \left(0, \frac{1}{2}\right), \quad S_3 = \left(0, \frac{1}{3}\right), \ldots.$$

In general, if n is a positive integer, then $S_n = \left(0, \dfrac{1}{n}\right)$.

Here's another collection of sets indexed by $\mathbb{N}$:

$$\begin{aligned}
S_1 &= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \\
S_2 &= \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\} \\
S_3 &= \{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\} \\
&\ \vdots
\end{aligned}$$

In general, S_n consists of the integers which are divisible by n.

Now let $I = \mathbb{R}$. Here's a collection of sets indexed by I:

$$S_x = \{x, -x\} \quad \text{for} \quad x \in \mathbb{R}.$$

For instance, I have sets S_3, $S_{-117/13}$, S_π, and so on, one for every real number.

Since $\mathbb{R}$ is **uncountable**, I can't *list* the sets in this collection the way I could list collections of sets indexed by $\mathbb{N}$.

Here are a couple of the sets:

$$S_{\sqrt{2}} = \{\sqrt{2}, -\sqrt{2}\}, \quad S_{42} = \{42, -42\}.$$

Here's another collection of sets indexed by $\mathbb{R}$:

$$S_x = [x, \infty) \quad \text{for} \quad x \in \mathbb{R}.$$

Each set in this collection is an interval consisting of all real numbers greater than or equal to x. So, for example,

$$S_1 = [1, \infty), \quad S_\pi = [\pi, \infty).$$

Definition. Let I be a set, and let $\{S_i\}$ be a collection of sets indexed by I.

(a) The **union** $\bigcup_{i \in I} S_i$ of the S_i is the set

$$\bigcup_{i \in I} S_i = \{s \mid s \in S_i \quad \text{for some} \quad i \in I\}.$$

(b) The **intersection** $\bigcap_{i \in I} S_i$ of the S_i is the set

$$\bigcap_{i \in I} S_i = \{s \mid s \in S_i \quad \text{for all} \quad i \in I\}.$$

Remark. For a collection of sets S_1, S_2, S_3, ... indexed by the natural numbers, you usually write the union and intersection this way:

$$\bigcup_{n=1}^{\infty} S_n \quad \text{and} \quad \bigcap_{n=1}^{\infty} S_n.$$

Example. Consider the following collection of sets indexed by $\mathbb{N}$:

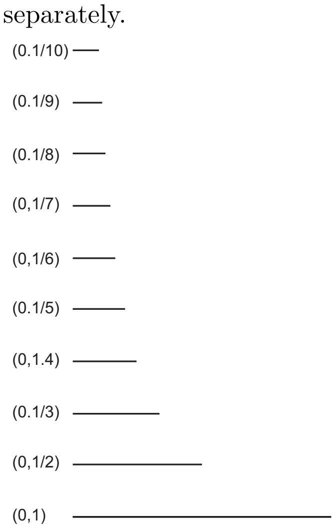
$$S_1 = (0, 1), \quad S_2 = \left(0, \frac{1}{2}\right), \quad S_3 = \left(0, \frac{1}{3}\right), \ldots, S_n = \left(0, \frac{1}{n}\right), \ldots.$$

Prove:

(a) $\displaystyle\bigcup_{n=1}^{\infty} \left(0, \frac{1}{n}\right) = (0, 1)$.

(b) $\displaystyle\bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right) = \emptyset$.

The collection of intervals is shown below. They actually lie on top of one another on the x-axis; I've "pulled them up" so you can see them separately.



(a) I will show each set is contained in the other. Let $x \in \bigcup_{n=1}^{\infty} \left(0, \frac{1}{n}\right)$. Then $x \in \left(0, \frac{1}{n}\right)$ for some $n > 1$. This means that $0 < x < \frac{1}{n}$.

Now $n > 1$ implies $\frac{1}{n} < 1$, so $0 < x < 1$. Hence, $x \in (0,1)$.

This proves that $\bigcup_{n=1}^{\infty} \left(0, \frac{1}{n}\right) \subset (0,1)$.

Conversely, suppose $x \in (0,1)$. Now $S_1 = (0,1)$, so by the definition of union, $x \in \bigcup_{n=1}^{\infty} \left(0, \frac{1}{n}\right)$. This proves that $(0,1) \subset \bigcup_{n=1}^{\infty} \left(0, \frac{1}{n}\right)$.

Hence, $\bigcup_{n=1}^{\infty} \left(0, \frac{1}{n}\right) = (0,1)$. $\square$

(b) Since the empty set is a subset of any set, I have $\emptyset \subset \bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right)$.

The opposite inclusion is $\bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right) \subset \emptyset$. To show this means to show that $\bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right)$ contains *no* elements. I'll give a proof by contradiction.

Suppose on the contrary that $c \in \bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right)$. By the definition of intersection, this means that $c \in \left(0, \frac{1}{n}\right)$ for every positive integer n.

Note that

$$\lim_{n \to \infty} \frac{1}{n} = 0.$$

In the limit definition, choose $\epsilon = c$. Then there is a number M such that for all $n > M$, I have

$$c = \epsilon > \left|\frac{1}{n} - 0\right| = \frac{1}{n}.$$

Choose a *positive integer* n such that $n > M$. Then

$$0 < \frac{1}{n} < c.$$

But this means that $c \notin \left(0, \frac{1}{n}\right)$, contradicting the fact that $c \in \left(0, \frac{1}{n}\right)$ for every positive integer n. This shows that there is no such element c, so the intersection is empty. $\square$

Example. Prove that $\bigcup_{n=1}^{\infty} \left[0, \frac{n}{n+1}\right] = [0,1)$.

First, I'll show that the left side is contained in the right side. Let $x \in \bigcup_{n=1}^{\infty} \left[0, \frac{n}{n+1}\right]$. I have to show that $x \in [0,1)$.

Since $x \in \bigcup_{n=1}^{\infty} \left[0, \frac{n}{n+1}\right]$, I know that $x \in \left[0, \frac{n}{n+1}\right]$ for some $n \geq 1$. This means that

$$0 \leq x \leq \frac{n}{n+1}.$$

But

$$\begin{aligned} 1 &> 0 \\ n+1 &> n \\ 1 &> \frac{n}{n+1} \end{aligned}$$

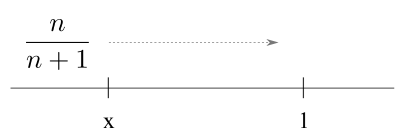
Therefore, $0 \le x < 1$. This means that $x \in [0, 1)$. Hence, $\bigcup_{n=1}^{\infty} \left[0, \frac{n}{n+1}\right] \subset [0, 1)$.

Next, I'll show that the right side is contained in the left side. Suppose $x \in [0, 1)$. I have to show that $x \in \bigcup_{n=1}^{\infty} \left[0, \frac{n}{n+1}\right]$.

Since $x \in [0, 1)$, I have $0 \le x < 1$. Note that

$$\lim_{n \to \infty} \frac{n}{n+1} = 1.$$

I'll pause to give a picture of what I'll do next. The idea is that since $\frac{n}{n+1}$ is approaching 1, and since $x < 1$, eventually the $\frac{n}{n+1}$ terms must become larger than x:



Intuitively, if all the $\frac{n}{n+1}$'s stayed to the left of x, then their limit couldn't be greater than x, so the limit couldn't be 1.

Continuing the proof, in the limit definition, let $\epsilon = 1 - x$. Then there is a number M such that if $n > M$,

$$1 - x = \epsilon > \left|\frac{n}{n+1} - 1\right|.$$

Since $\frac{n}{n+1} < 1$, the absolute value becomes

$$-\left(\frac{n}{n+1} - 1\right) = 1 - \frac{n}{n+1}.$$

The inequality above becomes

$$\begin{aligned} 1 - x &> 1 - \frac{n}{n+1} \\ -x &> -\frac{n}{n+1} \\ x &< \frac{n}{n+1} \end{aligned}$$

That is, for some n I have $x < \frac{n}{n+1}$. Since I already know $x \ge 0$, I have

$$0 \le x < \frac{n}{n+1}.$$

This means that $x \in \left[0, \frac{n}{n+1}\right)$. By the definition of union, $x \in \bigcup_{n=1}^{\infty} \left[0, \frac{n}{n+1}\right]$. Therefore, $[0, 1) \subset \bigcup_{n=1}^{\infty} \left[0, \frac{n}{n+1}\right]$.

Since I've proved both inclusions, I have $\displaystyle\bigcup_{n=1}^{\infty}\left[0, \frac{n}{n+1}\right] = [0,1)$. $\square$

Example. Prove that

$$\bigcap_{n=1}^{\infty}\left[1, 3+\frac{1}{n}\right] = [1,3].$$

I'll show that each of the sets $\displaystyle\bigcap_{n=1}^{\infty}\left[1, 3+\frac{1}{n}\right]$ and $[1,3]$ is contained in the other.

I'll do the easy inclusion first. Let $x \in [1,3]$. Then $1 \le x \le 3$.

For all $n \ge 1$, I have $3 < 3 + \dfrac{1}{n}$. Hence,

$$1 \le x \le 3 < 3 + \frac{1}{n}.$$

Therefore, $x \in \left[1, 3+\dfrac{1}{n}\right]$ for all $n \ge 1$. By definition of intersection, $x \in \displaystyle\bigcap_{n=1}^{\infty}\left[1, 3+\frac{1}{n}\right]$.

Thus, $[1,3] \subset \displaystyle\bigcap_{n=1}^{\infty}\left[1, 3+\frac{1}{n}\right]$.

Next, let $x \in \displaystyle\bigcap_{n=1}^{\infty}\left[1, 3+\frac{1}{n}\right]$. This means that $x \in \left[1, 3+\dfrac{1}{n}\right]$ for all $n \ge 1$ — that is,

$$1 \le x \le 3 + \frac{1}{n} \quad \text{for all} \quad n \ge 1.$$

I have to show that $x \le 3$. Suppose on the contrary that $x > 3$.
Note that

$$\lim_{n\to\infty}\left(3 + \frac{1}{n}\right) = 3.$$

In the limit definition, let $\epsilon = x - 3$. Then there is a number M such that if $n > M$,

$$\epsilon = x - 3 > \left|3 + \frac{1}{n} - 3\right| = \left|\frac{1}{n}\right| = \frac{1}{n}.$$

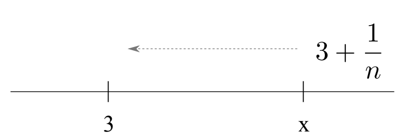
(I can drop the absolute values because n is positive.)
For any n such that $n > M$, I have

$$x - 3 > \frac{1}{n}$$

$$x > 3 + \frac{1}{n}$$

But this contradicts the fact that $1 \le x \le 3 + \dfrac{1}{n}$ for all $n \ge 1$.

Intuitively, since $\displaystyle\lim_{n\to\infty}\left(3 + \frac{1}{n}\right) = 3$, if $x > 3$ then eventually the $3 + \dfrac{1}{n}$'s must shrink to the left of x.



If all of them stayed to the right of x, the limit would be greater than or equal to x, so it couldn't be 3. This proves by contradiction that $x \le 3$. Since I already know that $1 \le x$, I have $1 \le x \le 3$, or $x \in [1, 3]$. Thus, $\displaystyle\bigcap_{n=1}^{\infty} \left[1, 3 + \frac{1}{n}\right] \subset [1, 3]$.

Together with the first inclusion, this proves that $\displaystyle\bigcap_{n=1}^{\infty} \left[1, 3 + \frac{1}{n}\right] = [1, 3]$. $\square$
