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Limits at Infinity

In this section, I'll discuss proofs for limits of the form $\lim_{x\to\infty} f(x)$. They are like ϵ-δ proofs, though the setup and algebra are a little different.

Recall that $\lim_{x\to c} f(x) = L$ means that for every $\epsilon > 0$, there is a δ such that if

$$\delta > |x - c| > 0, \quad \text{then} \quad \epsilon > |f(x) - L|.$$

Definition. $\lim_{x\to\infty} f(x) = L$ means that for every $\epsilon > 0$, there is an M such that if

$$x > M, \quad \text{then} \quad \epsilon > |f(x) - L|.$$

In other words, I can make $f(x)$ as close to L as I please by making x sufficiently large.

Remarks. Limits at infinity often occur as **limits of sequences**, such as

$$\frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \ldots, \frac{1}{n}, \ldots.$$

In this case, $\lim_{n\to\infty} \frac{1}{n} = 0$. I won't make a distinction between the limit at infinity of a sequence and the limit at infinity of a function; the proofs you do are essentially the same in both cases.

There is s similar definition for $\lim_{x\to-\infty} f(x) = L$, and the proofs are similar as well. I'll stick to $\lim_{x\to\infty} f(x)$ here.

Example. Prove that $\lim_{n\to\infty} \frac{1}{n} = 0$.

As with ϵ-δ proofs, I do some scratch work, working backwards from what I want. Then I write the "real proof" in the forward direction.

Scratch work. I want

$$\epsilon > \left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| = \frac{1}{n}.$$

I want to drop the absolute values, so I'll assume $n > 0$. Rearranging the inequality, I get $n > \frac{1}{\epsilon}$.

Here's the real proof. Let $\epsilon > 0$. Set $M = \frac{1}{\epsilon}$. Since $\epsilon > 0$, I have $M = \frac{1}{\epsilon} > 0$. Suppose $n > M$. Then $n > M > 0$, and

$$\begin{aligned} n &> M = \frac{1}{\epsilon} \\ \epsilon &> \frac{1}{n} \\ \epsilon &> \left|\frac{1}{n}\right| \\ \epsilon &> \left|\frac{1}{n} - 0\right| \end{aligned}$$

This proves that $\lim_{n\to\infty} \frac{1}{n} = 0$. $\square$

Example. Prove that $\lim\limits_{x\to\infty} \dfrac{6x+1}{2x+1} = 3$.

Scratch work. I want

$$\epsilon > \left|\frac{6x+1}{2x+1} - 3\right| = \left|\frac{6x+1-3(2x+1)}{2x+1}\right| = \left|\frac{-2}{2x+1}\right| = \left|\frac{2}{2x+1}\right| = \frac{2}{2x+1}.$$

In order to drop the absolute values, I need to assume $x > 0$.
Rearrange the inequality:

$$\begin{aligned} \epsilon &> \frac{2}{2x+1} \\ (2x+1)\epsilon &> 2 \\ 2x\epsilon + \epsilon &> 2 \\ 2x\epsilon &> 2 - \epsilon \\ x &> \frac{2-\epsilon}{2\epsilon} \end{aligned}$$

Here's the real proof. Let $\epsilon > 0$. Set $M = \max\left(0, \dfrac{2-\epsilon}{2\epsilon}\right)$. If $x > M$, then $x > 0$ and $x > \dfrac{2-\epsilon}{2\epsilon}$. So

$$\begin{aligned} x &> \frac{2-\epsilon}{2\epsilon} \\ 2\epsilon x &> 2 - \epsilon \\ 2\epsilon x + \epsilon &> 2 \\ \epsilon(2x+1) &> 2 \\ \epsilon &> \frac{2}{2x+1} \\ \epsilon &> \left|\frac{2}{2x+1}\right| \\ \epsilon &> \left|\frac{-2}{2x+1}\right| \\ \epsilon &> \left|\frac{6x+1-3(2x+1)}{2x+1}\right| \\ \epsilon &> \left|\frac{6x+1}{2x+1} - 3\right| \end{aligned}$$

Therefore,

$$\lim_{x\to\infty} \frac{6x+1}{2x+1} = 3. \quad \square$$

Note that the expression $\dfrac{2-\epsilon}{2\epsilon}$ would be negative if $\epsilon > 2$. So I took M to be the max of 0 and $\dfrac{2-\epsilon}{2\epsilon}$ to ensure that if $x > M$, then x would be positive. Now you actually need $2x+1$ to be positive in order to put on the absolute values, and $2x+1 > 0$ if $x > -\dfrac{1}{2}$. It isn't hard to prove that $\dfrac{2-\epsilon}{2\epsilon} > -\dfrac{1}{2}$, so in fact I don't need to take the max with 0 — provided that I'm willing to *prove* that $\dfrac{2-\epsilon}{2\epsilon} > -\dfrac{1}{2}$. I decided to take the easy way out!

Example. Prove that $\lim_{n\to\infty} (-1)^n$ is undefined.

I'll use proof by contradiction. Suppose that

$$\lim_{n\to\infty} (-1)^n = L.$$

Taking $\epsilon = \frac{1}{2}$ in the definition, I can find M such that if $n > M$, then $\frac{1}{2} > |(-1)^n - L|$. Choose p to be an even number greater than M. Then

$$\frac{1}{2} > |(-1)^p - L| = |1 - L|.$$

This says that the distance from L to 1 is less than $\frac{1}{2}$, so

$$\frac{1}{2} < L < \frac{3}{2}.$$

Choose q to be an odd number greater than M. Then

$$\frac{1}{2} > |(-1)^q - L| = |-1 - L|.$$

This says that the distance from L to -1 is less than $\frac{1}{2}$, so

$$-\frac{3}{2} < L < -\frac{1}{2}.$$

This is a contradiction, since L can't be in $\left(\frac{1}{2}, \frac{3}{2}\right)$ and in $\left(-\frac{3}{2}, -\frac{1}{2}\right)$ at the same time. Hence, $\lim_{n\to\infty} (-1)^n$ is undefined. $\square$