Limits

The definition of a **limit** involves both universal and existential quantifiers.

Let f be a function from the real numbers to the real numbers, and let c be a real number. Assume that f is defined on a open interval containing c. The statement $\lim_{x \to c} f(x) = L$ means:

For every $\epsilon > 0$, there is a $\delta > 0$, such that if $\delta > |x - c| > 0$, then $\epsilon > |f(x) - L|$.

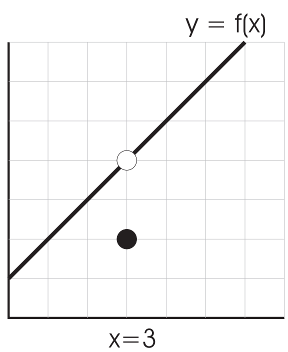
Think of δ as a thermostat, $f(x)$ as the actual temperature in a room, and L as the ideal temperature. Someone challenges you to make the actual temperature $f(x)$ fall within a certain tolerance ϵ of the ideal temperature L. You must do that by setting your δ-thermostat appropriately (so that x is sufficiently close to c).

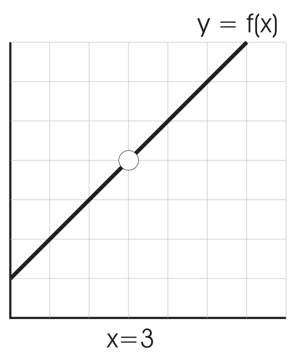
Moreover, note that it says "*for every* $\epsilon > 0$". It's isn't enough for you to say what you'd do if you were challenged with $\epsilon = 0.1$ or $\epsilon = 0.000004$. You must prove that you can meet the challenge *no matter what* ϵ *you're challenged with*.

Finally, note the stipulation "$|x - c| > 0$". This implies that $x \neq c$, since $x = c$ gives $|x - c| = 0$. Thus, the conclusion "$\epsilon > |f(x) - L|$" *must* hold only for x's *close to* c, but not necessarily for $x = c$. (It *may* hold for $x = c$, but it doesn't *have to*.)

What does this mean? It's a precise way of saying that the value of the limit of $f(x)$ as x approaches c does not depend on what $f(x)$ does *at* $x = c$ — over even whether $f(c)$ is defined.

For example, consider the functions whose graphs are shown below.





In both cases,

$$\lim_{x \to 3} f(x) = 4.$$

In the first case, $f(3) = 2$: The value of the function at $x = 3$ is different from the value of the limit.

In the second case, $f(3)$ is undefined.

The fact that $\lim_{x \to 3} f(x) \neq f(3)$ means that f is *not continuous* at $x = 3$.

Example. Use the ϵ-δ definition of the limit to prove that

$$\lim_{x \to 2} (5x + 4) = 14.$$

In this case, $c = 2$, $f(x) = 5x + 4$, and $L = 14$. So here is what I need to prove.

Suppose $\epsilon > 0$. I must find a $\delta > 0$ such that if $\delta > |x - 2| > 0$, then $\epsilon > |(5x + 4) - 14|$.

Note that at this point ϵ is fixed — given — but all you can assume is that it's some positive number. Since it *is* given, however, I can use it in finding an appropriate δ.

I'll show how to find δ by working backwards; then I'll write the proof "forwards", the way you should write it.

I want

$$\epsilon > |(5x+4)-14|, \quad \text{or} \quad \epsilon > |5x-10|, \quad \text{or} \quad \frac{\epsilon}{5} > |x-2|.$$

It looks like I should set $\delta = \dfrac{\epsilon}{5}$.

All of this has been on "scratch paper"; now here's the real proof.

Suppose $\epsilon > 0$. Let $\delta = \dfrac{\epsilon}{5}$. If $\delta > |x-2| > 0$, then

$$\frac{\epsilon}{5} > |x-2|, \quad \text{so} \quad \epsilon > |5x-10|, \quad \text{or} \quad \epsilon > |(5x+4)-14|.$$

Thus, if $\delta = \dfrac{\epsilon}{5}$ and $\delta > |x-2| > 0$, then $\epsilon > |(5x+4)-14|$. This proves that $\displaystyle\lim_{x\to 2}(5x+4) = 14$. $\square$

Example. Let

$$f(x) = \begin{cases} 3x+4 & \text{if } x < 1 \\ 9-2x & \text{if } x \ge 1 \end{cases}.$$

Use the ϵ-δ definition of the limit to prove that

$$\lim_{x\to 1} f(x) = 7.$$

Let $\epsilon > 0$. I must find $\delta > 0$ such that if $\delta > |x-1| > 0$, then $\epsilon > |f(x)-7|$.

Here's my scratch work. First, for $x < 1$,

$$\epsilon > |f(x)-7|, \quad \epsilon > |(3x+4)-7|, \quad \epsilon > |3x-3|, \quad \frac{\epsilon}{3} > |x-1|.$$

It looks like I should take $\delta = \dfrac{\epsilon}{3}$.

For $x > 1$,

$$\epsilon > |f(x)-7|, \quad \epsilon > |(9-2x)-7|, \quad \epsilon > |2-2x| = |2x-2|, \quad \frac{\epsilon}{2} > |x-1|.$$

It looks like I should take $\delta = \dfrac{\epsilon}{2}$.

In order to ensure that both the $x < 1$ and $x > 1$ requirements are satisfied, I'll take δ to be the smaller of the two: $\delta = \dfrac{\epsilon}{3}$.

Now here's the proof written out correctly.

Suppose $\epsilon > 0$. Let $\delta = \dfrac{\epsilon}{3}$, and assume that $\delta > |x-1| > 0$.

If $x < 1$, then

$$\frac{\epsilon}{3} > |x-1|, \quad \text{so} \quad \epsilon > |3x-3| = |(3x+4)-7| = |f(x)-7|.$$

Now consider the case $x > 1$. Since $\dfrac{\epsilon}{3} > |x-1|$, and since $\dfrac{\epsilon}{2} > \dfrac{\epsilon}{3}$, I have $\dfrac{\epsilon}{2} > |x-1|$. Therefore,

$$\epsilon > |2x-2| = |2-2x| = |(9-2x)-7| = |f(x)-7|.$$

(The case $x = 1$ is ruled out because $|x - 1| > 0$.)

Thus, taking $\delta = \dfrac{\epsilon}{3}$ guarantees that if $\delta > |x-1| > 0$, then $\epsilon > |f(x)-7|$. This proves that $\lim_{x \to 1} f(x) = 7$.

□

Example. Use the ϵ-δ definition of the limit to prove that

$$\lim_{x \to 2} x^2 = 4.$$

Let $\epsilon > 0$. I want to find $\delta > 0$ such that if $\delta > |x - 2| > 0$, then $\epsilon > |x^2 - 4|$.

I start out as usual with my scratch work:

$$\epsilon > |x^2 - 4| = |x - 2||x + 2|.$$

Now I have a problem. I can use δ to control $|x - 2|$, but what do I do about $|x + 2|$?

The idea is this: Since I have complete control over δ, I can *assume* $\delta \le 1$. When I finally set δ, I can make it smaller if necessary to ensure that this condition is met.

Now if $\delta \le 1$, then $|x - 2| < 1$, so $1 < x < 3$, and $3 < x + 2 < 5$. In particular, the *biggest* $|x + 2|$ could be is 5. So now

$$\epsilon > |x - 2||x + 2| \quad \text{becomes} \quad \epsilon > |x - 2| \cdot 5, \quad \text{so} \quad \frac{\epsilon}{5} > |x - 2|.$$

This inequality suggests that I set $\delta = \dfrac{\epsilon}{5}$ — but then I remember that I needed to assume $\delta \le 1$. I can meet both of these conditions by setting δ to *the smaller of* 1 and $\dfrac{\epsilon}{5}$: that is, $\delta = \min\left(1, \dfrac{\epsilon}{5}\right)$.

That was scratchwork; now here's the real proof.

Let $\epsilon > 0$. Set $\delta = \min\left(1, \dfrac{\epsilon}{5}\right)$. Suppose $\delta > |x - 2| > 0$.

Since $\delta \le 1$, I have

$$\begin{aligned} 1 &> |x - 2| \\ 1 < x &< 3 \\ 3 < x + 2 &< 5 \end{aligned}$$

Therefore, $5 > |x + 2|$.

Now $\delta \le \dfrac{\epsilon}{5}$, so $\dfrac{\epsilon}{5} > |x - 2|$.

Now multiply the inequalities $5 > |x + 2|$ and $\dfrac{\epsilon}{5} > |x - 2|$:

$$\epsilon = \frac{\epsilon}{5} \cdot 5 > |x - 2||x + 2| = |x^2 - 4|.$$

Thus, if $\delta = \min\left(1, \dfrac{\epsilon}{5}\right)$ and $\delta > |x - 2| > 0$, then $\epsilon > |x^2 - 4|$. This proves that $\lim_{x \to 2} x^2 = 4$. □

Example. Prove that $\displaystyle \lim_{x \to 2} \frac{x^2 + 11}{x + 3} = 3$.

Let $\epsilon > 0$. I must find δ such that if $\delta > |x - 2| > 0$, then $\epsilon > \left|\dfrac{x^2 + 11}{x + 3} - 3\right|$.

I'll start with some scratchwork.

$$\left|\frac{x^2 + 11}{x + 3} - 3\right| = \left|\frac{x^2 + 11 - 3(x + 3)}{x + 3}\right| = \left|\frac{x^2 - 3x + 2}{x + 3}\right| = \left|\frac{(x - 2)(x - 1)}{x + 3}\right| = |x - 2|\left|\frac{x - 1}{x + 3}\right|.$$

I can use δ to control $|x-2|$ directly. I need to control the size of $\left|\dfrac{x-1}{x+3}\right|$. It's important to think of this as $|x-1|\cdot\left|\dfrac{1}{x+3}\right|$, *not* as $|x-1|$ and $|x+3|$!

Assume $1 \ge \delta$. Then $1 > |x-2|$, so $1 < x < 3$.

For $x-1$, $0 < x-1 < 2$, so $|x-1| < 2$.

For $\left|\dfrac{1}{x+3}\right|$, $4 < x+3 < 6$, so $\dfrac{1}{4} > \dfrac{1}{x+3} > \dfrac{1}{6}$, and $\left|\dfrac{1}{x+3}\right| < \dfrac{1}{4}$.

Since all the number involved are positive, I can multiply the inequalities to obtain

$$2\cdot\frac{1}{4} > |x-1|\cdot\left|\frac{1}{x+3}\right|, \quad \text{or} \quad \frac{1}{2} > |x-1|\cdot\left|\frac{1}{x+3}\right|.$$

Thus, I'll get $\epsilon > |x-2|\left|\dfrac{x-1}{x+3}\right|$ if I have $\epsilon > |x-2|\cdot\dfrac{1}{2}$, or $2\epsilon > |x-2|$. Here's the proof.

Let $\epsilon > 0$. Set $\delta = \min(2\epsilon, 1)$. Suppose $\delta > |x-2| > 0$.

Since $1 \ge \delta$, $1 > |x-2|$, and $1 < x < 3$.

First, $0 < x-1 < 2$, so $|x-1| < 2$.

Next, $\left|\dfrac{1}{x+3}\right|$, $4 < x+3 < 6$, so $\dfrac{1}{4} > \dfrac{1}{x+3} > \dfrac{1}{6}$, and $\left|\dfrac{1}{x+3}\right| < \dfrac{1}{4}$.

Hence,

$$2\cdot\frac{1}{4} > |x-1|\cdot\left|\frac{1}{x+3}\right|, \quad \text{or} \quad \frac{1}{2} > |x-1|\cdot\left|\frac{1}{x+3}\right|.$$

In addition, $2\epsilon \ge \delta > |x-2|$. Therefore,

$$\epsilon > |x-2|\cdot\frac{1}{2} > |x-2|\cdot|x-1|\cdot\left|\frac{1}{x+3}\right| = \left|\frac{x^2+11}{x+3} - 3\right|.$$

This proves that $\displaystyle\lim_{x\to 2}\frac{x^2+11}{x+3} = 3$. $\square$
