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Logical Connectives

Mathematics works according to the laws of logic, which specify how to make valid deductions. In order to apply the laws of logic to mathematical statements, you need to understand their *logical forms*.

If you take a course in mathematical logic, you will see a formal discussion of proofs. You start with a **formal language**, which describes the symbols you're allowed to use and how to combine them, and **rules of inference**, which describe the valid ways of making steps in a proof. Everything is symbols! If I did that here you would probably find it hard to follow. So the discussion here will be informal (though you might not think it is!).

Proofs are composed of **statements**. A **statement** is a declarative sentence that can be either true or false.

Remark. Many real proofs contain things which aren't really statements — questions, descriptions, and so on. They're there to help to explain things for the reader. When I say "Proofs are composed of statements", I'm referring to the actual mathematical content with the explanatory material removed.

Example. Which of the following are statements? If it is a statement, determine if possible whether it's true or false.

"Calvin Butterball is a math major."

"$0 = 1$."

"The diameter of the earth is 1 inch or I ate a pizza."

"Do you have a pork barbecue sandwich?"

"Give me a cafe mocha!"

"$1 + 1 = 2$."

"$1 + 1$."

"Calvin Butterball is a math major" is a statement. You'd need to know more about Calvin and math majors to know whether the statement is true or false.

"$0 = 1$" is a statement which is false (assuming that "0" and "1" refer to the real numbers 0 and 1).

"The diameter of the earth is 1 inch or I will have a pizza" is a statement. The first part ("The diameter of the earth is 1 inch") is false, but you would need to know something about my recent meals to know whether "I ate a pizza" is true or false. Nevertheless, it's reasonable to suppose that you could figure out whether "I ate a pizza" is true or false — and hence, whether the original "or" statement is true or false.

"Do you have a pork barbecue sandwich?" is not a statement — it's a question.

Likewise, "Give me a cafe mocha!" is not a statement — it's an imperative sentence, i.e. an order to do something.

"$1+1 = 2$" is a statement. An easy way to tell is to *read it* and see if it's a *complete declarative sentence* which is either true or false. This statement would read (in words):

"One plus one equals two."

You can see that it's a complete declarative sentence (and it happens to be a true statement about real numbers).

On the other hand, "$1+1$" is *not* a statement. It would be read "One plus one", which is not a sentence since it doesn't have a verb. (Things like "$1+1$" are referred to as **terms** or **expressions**.)

Since proofs are composed of **statements**, you should never have isolated phrases (like $1+1$ or "$(a+b)^2$") in your proofs. Be sure that every line of a proof is a statement. Read each line to yourself to be sure. $\square$

In terms of logical form, statements are built from simpler statements using **logical connectives**.

Definition. The logical connectives of sentential logic are:

(a) **Negation** ("not"), denoted $\neg$.

(b) **Conjunction** ("and"), denoted $\wedge$.

(c) **Disjunction** ("or"), denoted $\vee$.

(d) **Conditional** ("if-then" or "implication"), denoted $\rightarrow$.

(e) **Biconditional** ("if and only if" or "double implication"), denoted $\leftrightarrow$.

Later I'll discuss the **quantifiers** "for all" (denoted $\forall$) and "there exists" (denoted $\exists$).

Remark. You may see different symbols used by other people. For example, some people use $\sim$ for negation. And $\Rightarrow$ is sometimes used for the conditional, in which case $\Leftrightarrow$ is used for the biconditional.

Example. Represent the following statements using logical connectives.

(a) P or not Q.

(b) If P and R, then Q.

(c) P if and only if (Q and R).

(d) Not P and not Q.

(e) It is not the case that if P, then Q.

(f) If P and Q, then R or S.

(a)
$$P \vee \neg Q \quad \square$$

(b)
$$(P \wedge R) \rightarrow Q \quad \square$$

(c)
$$P \leftrightarrow (Q \wedge R) \quad \square$$

(d)
$$\neg P \wedge \neg Q \quad \square$$

(e)
$$\neg(P \rightarrow Q) \quad \square$$

(f)
$$(P \wedge Q) \rightarrow (R \vee S) \quad \square$$

Remark. You might object that (for instance) "$P \vee Q$", which you would read as "P or Q" does not seem like a statement (a complete English sentence). However, in the context of a proof, the symbols P and Q would stand for statements, and replacing P and Q with the statements they stand for result in a complete English sentence (for example, "The diameter of the earth is 1 inch or I ate a pizza").

Other words or phrases may occur in statements. Here's a table of some of them and how they are translated.

Phrase	**Logical translation**
P, but Q	$P \wedge Q$
Either P or Q	$P \vee Q$
P or Q, but not both	$(P \vee Q) \wedge \neg(P \wedge Q)$
P if Q	$Q \rightarrow P$
P is necessary for Q	$Q \rightarrow P$
P is sufficient for Q	$P \rightarrow Q$
P only if Q	$P \rightarrow Q$
P is equivalent to Q	$P \leftrightarrow Q$
P whenever Q	$Q \rightarrow P$

Consider the word "but", for example. If I say "Calvin is here, but Bonzo is there", I mean that Calvin is here *and* Bonzo is there. My intention is that both of the statements should be true. That is the same as what I mean when I say "Calvin is here and Bonzo is there".

In practice, mathematicians tend to use a small set of phrases over and over. It doesn't make for exciting reading, but it allows the reader to concentrate on the meaning of what is written. For example, a mathematician will usually say "if Q, then P", rather than the logically equivalent "P whenever Q" or "P only if Q". The last two versions are less familiar, and so it make take a reader longer to figure out what is meant.

In my opinion, *you should avoid the expressions in the table above in writing math.* Keep things boring and simple.

This is a good time to discuss the way the word "or" is used in mathematics. When you say "I'll have dinner at MacDonald's or at Pizza Hut", you *probably* mean "or" in its **exclusive** sense: You'll have dinner at MacDonald's *or* you'll have dinner at Pizza Hut, *but not both.* The "but not both" is what makes this an **exclusive or**.

Mathematicians use "or" in the **inclusive** sense. When "or" is used in this way, "I'll have dinner at MacDonald's or at Pizza Hut" means you'll have dinner at MacDonald's *or* you'll have dinner at Pizza Hut, *or possibly both.* Obviously, I'm not *guaranteeing* that both will occur; I'm just not ruling it out.

The reason for this choice is probably that, when the word "or" comes up in math, it usually comes up in an inclusive way. For example, if X and Y are sets, their union consists of things which are in X or in Y *or in both.* So if we chose to use "or" in the exclusive way, I have to say "or in both". With the inclusive "or", I can just say "in X or in Y", since the "in both" is assumed. As with many conventions in math, it's the way it is because we're lazy and it saves writing.

Example. Translate the following statements into logical notation, using the following symbols:

S = "The stromboli is hot."

L = "The lasagne is cold."

P = "The pizza will be delivered."

(a) "The stromboli is hot and the pizza will not be delivered."

(b) "If the lasagne is cold, then the pizza will be delivered."

(c) "Either the lasagne is cold or the pizza won't be delivered."

(d) "If the pizza won't be delivered, then both the stromboli is hot and the lasagne is cold."

(e) "The lasagne isn't cold if and only if the stromboli isn't hot."

(f) "The pizza will be delivered only if the lasagne is cold."

(g) "The stromboli is hot and the lasagne isn't cold, but the pizza will be delivered."

(a)
$$S \wedge \neg P \quad \square$$

(b)
$$L \to P \quad \square$$

(c)
$$L \vee \neg P \quad \square$$

(d)
$$\neg P \to (S \wedge L) \quad \square$$

(e)
$$\neg L \leftrightarrow \neg S \quad \square$$

(f)
$$P \to L \quad \square$$

(g)
$$S \wedge \neg L \wedge P \quad \square$$

The order of precedence of the logical connectives is:

1. Negation
2. Conjunction
3. Disjunction
4. Implication
5. Double implication

As usual, parentheses override the other precedence rules.

In most cases, it's best for the sake of clarity to use parentheses even if they aren't required by the precedence rules. For example, it's better to write

$$(P \wedge Q) \vee R \quad \text{rather than} \quad P \wedge Q \vee R.$$

Precedence would group P and Q anyway, but the first expression is clearer.

It's not common practice to use parentheses for grouping in ordinary sentences. Therefore, when you're translating logical statements into words, you may need to use certain expressions to indicate grouping.

(a) The combination "Either ... or ..." is used to indicate that everything between the "either" and the "or" is the first part of the "or" statement.

(b) The combination "Both ... and ..." is used to indicate that everything between the "both" and the "and" is the first part of the "and" statement.

In some cases, the only way to say something unambiguously is to be a bit wordy. Fortunately, mathematicians find ways to express themselves which are clear, yet avoid excessive linguistic complexity.

Example. Suppose that

C = "The cheesesteak is good."

F = "The french fries are greasy."

W = "The wings are spicy."

Translate the following logical statements into words (with no logical symbols):

(a) $(\neg C \wedge F) \rightarrow W$

(b) $\neg(C \vee W)$

(c) $\neg(\neg W \wedge C)$

(d) $\neg(\neg F)$.

(a) "If the cheesesteak isn't good and the french fries are greasy, then the wings are spicy." $\square$

(b) If I say "It's not the case that the cheesesteak is good or the wings are spicy", it might not be clear whether the negation applies only to "the cheesesteak is good" or to the disjunction "the cheesesteak is good or the wings are spicy".

So it's better to say "It's not the case that either the cheesesteak is good or the wings are spicy", since the "either" implies that "the cheesesteak is good" or "the wings are spicy" are *grouped together* in the or-statement.

In this case, the "either" blocks the negation from applying to "the cheesesteak is good", so the negation has to apply to the whole "or" statement. $\square$

(c) "It's not the case that both the wings aren't spicy and the cheesesteak is good." As with the use of the word "either" in (b), I've added the word "both" to indicate that the initial negation applies to the conjunction "the wings aren't spicy and the cheesesteak is good".

In this case, the "both" blocks the negation from applying to "the wings aren't spicy", so the negation has to apply to the whole "and" statement. $\square$

(d) The literal translation is "It's not the case that the french fries aren't greasy". Or (more awkwardly) you could say "It's not the case that it's not the case that the french fries are greasy".

Of course, this means the same thing as "The french fries are greasy".

To answer this kind of question, you should probably ask whether it's to be translated literally by symbols ("syntactically") or by meaning ("semantically"). In practice, mathematicians would almost always simplify to remove a double negation. $\square$

Our earlier examples have used real-world statements. What about actual mathematics?

Example. Express the following examples of actual mathematical text using logical symbols. (You do not need to know what these statements are talking about!)

(a) ([1], Theorem 25.11) In the semi-simple ring R, let $L = Re$ be a left ideal with generating idempotent e. Then L is a minimal left ideal if and only if eRe is a skew field.

(b) ([2], Proposition 14.11) Let X and Y be CW-complexes. Then $X \times Y$ (with the compactly generated topology) is a CW complex, and $X \vee Y$ is a subcomplex.

(The numbers in square brackets are *references* (like foonotes). I'll say something about them at the end of this section.)

(a) You could express this using logical connectives in the following way. Let

A = "R is a semi-simple ring".

B = "$L = Re$ is a left ideal with generating idempotent e".

C = "L is a minimal left ideal".

D = "eRe is a skew field".

The statement can be translated as $(A \wedge B) \to (C \leftrightarrow D)$.

Notice that to determine the *logical form*, you don't have to know what the words mean!

Mathematicians use the word "let" to introduce *hypotheses* in the statement of a theorem. From the point of view of logical form, the statements that accompany "let" form the if-part of a conditional, as statements A and B do here. □

(b) Let

P = "X and Y are CW-complexes".

Q = "$X \times Y$ (with the compactly generated topology) is a CW complex".

R = "$X \vee Y$ is a subcomplex".

The proposition can then be written in logical notation as $P \to (Q \wedge R)$.

Notice that you can often translate a statement in different ways. For example, I could have let

A = "X is a CW-complex".

B = "Y is a CW-complex".

C = "$X \times Y$ (with the compactly generated topology) is a CW complex".

D = "$X \vee Y$ is a subcomplex".

Now the proposition becomes $(A \wedge B) \to (C \wedge D)$. There is no difference in mathematical content, and no difference in terms of how you would prove it. □

Note: It would be better to express the statements above using **quantifiers**, which we will discuss later.

By the way, mathematicians usually do *not* translate mathematical statements into logical notation in *doing* mathematics (unless they happen to be working the in area of mathematical logic). Logical formalism serves as a foundation for math — and when, at the start, you get confused about a point of logic, it can be helpful to think of things in terms of logical notation. After a point, mathematicians gain an intuitive sense for correct logic as it is needed for their work. Then translating the math into logic is just extra work and can make things harder to comprehend.

We learn how to read, write, and speak using *words* early in our lives. It's the natural way for people to communicate and to understand. Symbols *compress* a log of meaning into small spaces, so they are good for short bursts of computation. But like packing lots of stuff in a box, compression hides meaning, which is why wading through solid pages of symbols in math papers can be so intimidating.

It's true that some people have suggested that mathematics *should* be written in a more formal way (for instance, to facilitate computer-aided proofs), but at the moment math is generally written in a combination of words and symbols. You should learn to use both appropriately.

As the last example shows, logical implications often arise in mathematical statements.

Definition. If $P \to Q$ is an implication, then:

(a) P is the **antecedent** or **hypothesis** and Q is the **consequent** or **conclusion**.

(b) The **converse** is the conditional $Q \to P$.

(c) The **inverse** is the conditional $\neg P \to \neg Q$.

(d) The **contrapositive** is the conditional $\neg Q \to \neg P$.

I will often use "if-part" instead of "antecedent" or "hypothesis", and "then-part" instead of "consequent" or "conclusion". While the terms in the definition are more traditional, I think "if-part" and "then-part" are clearer for people nowadays.

Example. Find the antecedent (if-part) and the consequent (then-part) of the following conditional statement:

"If $x > y$, then $y >$ Calvin."

Construct the converse, the inverse, and the contrapositive.

The antecedent is "$x > y$" and the consequent is "$y >$ Calvin".
The converse is "If $y >$ Calvin, then $x > y$".
The inverse is "If $x \not> y$, then $y \not>$ Calvin."
The contrapositive is "If $y \not>$ Calvin, then $x \not> y$".
Later on, I'll show that a conditional statement and its contrapositive are **logically equivalent**. □

Example. Construct the converse, the inverse, and the contrapositive of the following conditional statement:

"If Calvin gets a hot dog, then Calvin doesn't get a soda."

The converse is "If Calvin doesn't get a soda, then Calvin gets a hot dog".
The inverse is "If Calvin doesn't get a hot dog, then Calvin gets a soda". (Note that the literal negation of the consequent is "It is not the case that Calvin doesn't get a soda". But the two negations cancel out — this is called **double negation** — so I get "Calvin gets a soda".)
The contrapositive is "If Calvin gets a soda, then Calvin doesn't get a hot dog". □

Different fields use different formats for citing sources. For instance, you may have seen books which referred to sources using footnotes. Mathematicians often use numbers in square brackets (like "[1]" or "[2]") for citations. The numbers refer to the references, which are listed at the end of the paper or book. Among other things, it makes for less clutter on the text pages, and is easier to typeset.

Some authors prefer to use abbreviations involving the name of the author or the date of publication instead of numbers. If you're publishing an article in a journal, the journal will usually have a style that you're expected to follow. I think the important things are to give complete references so that a reader can look them up, and to make it clear what reference in the bibliography you're referring to.

Here are the references which I cited in the example above.

[1] Charles W. Curtis and Irving Reiner, *Representation theory of finite groups and associative algebras.* New York: Interscience Publishers, 1962. [ISBN 0-470-18975-4]

[2] Brayton Gray, *Homotopy theory.* New York: Academic Press, 1975. [ISBN 0-12-296050-5]