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Order Relations

A **partial order** on a set is, roughly speaking, a relation that behaves like the relation $\leq$ on $\mathbb{R}$.

Definition. Let X be a set, and let $\sim$ be a relation on X. $\sim$ is a **partial order** if:

(a) (Reflexive) For all $x \in X$, $x \sim x$.

(b) (Antisymmetric) For all $x, y \in X$, if $x \sim y$ and $y \sim x$, then $x = y$.

(c) (Transitive) For all $x, y, z \in X$, if $x \sim y$ and $y \sim z$, then $x \sim z$.

Example. For each relation, check each axiom for a partial order. If the axiom holds, prove it. If the axiom does not hold, give a specific counterexample.

(a) The relation $\leq$ on $\mathbb{R}$.

(b) The relation $<$ on $\mathbb{R}$.

(a) For all $x \in \mathbb{R}$, $x \leq x$: Reflexivity holds.

For all $x, y \in \mathbb{R}$, if $x \leq y$ and $y \leq x$, then $x = y$: Antisymmetry holds.

For all $x, y, z \in \mathbb{R}$, if $x \leq y$ and $y \leq z$, then $x \leq z$: Transitivity holds.

Thus, $\leq$ is a partial order. $\square$

(b) For no x is it true that $x < x$, so reflexivity fails.

Antisymmetry would say: If $x < y$ and $y < x$, then $x = y$. However, for no $x, y \in \mathbb{R}$ is it true that $x < y$ *and* $y < x$. Therefore, the first part of the conditional is false, and the conditional is true. Thus, antisymmetry is *vacuously true*.

If $x < y$ and $y < z$, then $x < z$. Therefore, transitivity holds.

Hence, $<$ is not a partial order. $\square$

Example. Let X be a set and let $\mathcal{P}(X)$ be the power set of X — i.e. the set of all subsets of X. Show that the relation of **set inclusion** is a partial order on $\mathcal{P}(X)$.

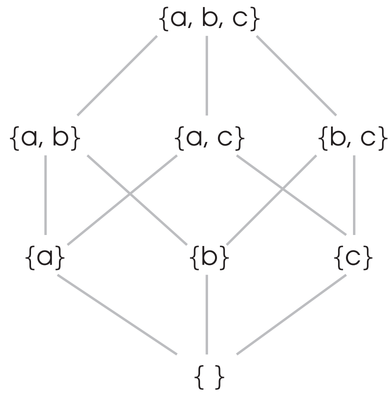
Subsets A and B of X are related under set inclusion if $A \subset B$.

If $A \subset X$, then $A \subset A$. The relation is reflexive.

Suppose $A, B \subset X$. If $A \subset B$ and $B \subset A$, then by definition of set equality, $A = B$. The relation is symmetric.

Finally, suppose $A, B, C \subset X$. If $A \subset B$ and $B \subset C$, then $A \subset C$. (You can write out the easy proof using elements.) The relation is transitive.

Here's a particular example. Let $X = \{a, b, c\}$. This is a picture of the set inclusion relation on $\mathcal{P}(X)$:



$\square$

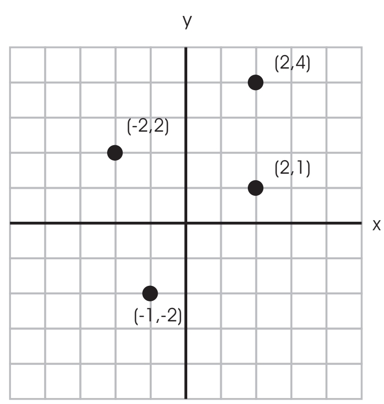
Definition. Let $(X, \leq)$ be a partially ordered set. The **lexicographic order** (or **dictionary order**) on $X \times X$ is defined as follows: $(x_1, y_1) \sim (x_2, y_2)$ means that

(a) $x_1 < x_2$, or

(b) $x_1 = x_2$ and $y_1 \leq y_2$.

Note that $(x_1, y_1) \sim (x_2, y_2)$ implies $x_1 \leq x_2$.

You can extend the definition to two different partially ordered sets X and Y, or a sequence X_1, X_2, $\ldots$, X_n of partially ordered sets in the same way. The name *dictionary order* comes from the fact that it describes the way words are ordered alphabetically in a dictionary. For instance, "aardvark" comes before "banana" because "a" comes before "b". If the first letters are the same, as with "mystery" and "meat", then you look at the second letters: "e" comes before "y", so "meat" comes before "mystery".



In the picture above, $(-2, 2) \sim (-1, -2)$, because $-2 < -1$. And $(2, 1) \sim (2, 4)$ because the x-coordinates are equal and $1 < 4$.
Proposition. The lexicographic order on $X \times X$ is a partial order.

Proof. First, $(x, y) \sim (x, y)$, since $x = x$ and $y \leq y$. $\sim$ is reflexive.
Next, suppose $(x_1, y_1) \sim (x_2, y_2)$ and $(x_2, y_2) \sim (x_1, y_1)$. Now $(x_1, y_1) \sim (x_2, y_2)$ means that either $x_1 < x_2$ or $x_1 = x_2$. The first case $x_1 < x_2$ is impossible, since this would contradict $(x_2, y_2) \sim (x_1, y_1)$. Therefore, $x_1 = x_2$. Then $(x_1, y_1) \sim (x_2, y_2)$ implies $y_1 \leq y_2$ and $(x_2, y_2) \sim (x_1, y_1)$ implies $y_2 \leq y_1$. Hence, $y_1 = y_2$. Therefore, $(x_1, y_1) = (x_2, y_2)$. $\sim$ is antisymmetric.
Finally, suppose $(x_1, y_1) \sim (x_2, y_2)$ and $(x_2, y_2) \sim (x_3, y_3)$. To keep things organized, I'll consider the four cases.

(a) If $x_1 < x_2$ and $x_2 < x_3$, then $x_1 < x_3$, so $(x_1, y_1) \sim (x_3, y_3)$.

(b) If $x_1 < x_2$ and $x_2 = x_3$, then $x_1 < x_3$, so $(x_1, y_1) \sim (x_3, y_3)$.

(c) If $x_1 = x_2$ and $x_2 < x_3$, then $x_1 < x_3$, so $(x_1, y_1) \sim (x_3, y_3)$.

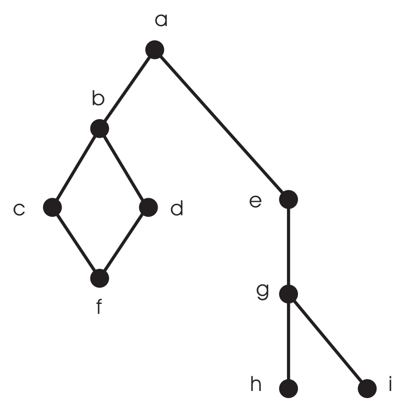
(d) If $x_1 = x_2$ and $x_2 = x_3$, then $y_1 \le y_2$ and $y_2 \le y_3$. This implies $x_1 = x_3$ and $y_1 \le y_3$, so $(x_1, y_1) \sim (x_3, y_3)$.

Hence, $\sim$ is transitive, and this completes the proof that $\sim$ is a partial order. $\square$

A common mistake in working with partial orders — and in real life — consists of *assuming* that if you have two things, then one must be bigger than the other. When this *is* true about two things, the things are said to be **comparable**. However, in an arbitrary partially ordered set, some pairs of elements are comparable and some are not.

Definition. Let $\sim$ be a relation on a set X. x and y in X are **comparable** if either $x \sim y$ or $y \sim x$.

Here's a pictorial example to illustrate the idea. You can sometimes describe an order relation by drawing a graph like the one below:



This picture shows a relation $\sim$ on the set

$$S = \{a, b, c, d, e, f, g, h, i\}.$$

Two elements are comparable if they're joining by a sequence of edges that goes upward "without reversing direction". (Think of "bigger" elements being above and "smaller" elements being below.) It's also understood that every element satisfies $x \sim x$.

For example, $f \sim c$, since there's an upward segment connecting f to c. And $f \sim a$, since there's an upward path of segments $f \to c \to b \to a$ connecting f to a.

On the other hand, there are elements which are not comparable. For example, d and e are not comparable, because there is no upward path of segments connecting one to the other. Likewise, $g \sim h$ and $g \sim i$, but h and i are not comparable.

Notice that a is comparable to every element of the set, and that $x \sim a$ for all $x \in S$.

Definition. Let X be a partially ordered set.

(a) An element $x \in X$ which is comparable to every other element of X and satisfies $x \ge y$ for all $y \in X$ is the **largest element** of the set.

(b) An element $x \in X$ which is comparable to every other element of X and satisfies $x \le y$ for all $y \in X$ is the **smallest element** of the set.

In some cases, we only care that an element be "bigger than" or "smaller than" elements to which it is comparable.

Definition. Let X be a partially ordered set. If an element x satisfies $x \geq y$ for all y to which it is comparable, then x is a **maximal element**. Likewise, if an element x satisfies $x \leq y$ for all y to which it is comparable, then x is a **minimal element**.

Note that a largest or smallest element, if it exists, is unique. On the other hand, there may be many maximal or minimal elements.

Example. Define a relation $\sim$ on $\mathbb{R}$ by

$$x \sim y \quad \text{means} \quad x^3 - 4x \leq y^3 - 4y.$$

Check each axiom for a partial order. If the axiom holds, prove it. If the axiom does not hold, give a specific counterexample.

$x^3 - 4x \leq x^3 - 4x$ for all $x \in \mathbb{R}$, so $x \sim x$ for all $x \in \mathbb{R}$. Therefore, $\sim$ is reflexive.

Suppose $x \sim y$ and $y \sim x$. Is is true that $x = y$?

$2 \sim -2$, since $2^3 - 4 \cdot 2 \leq (-2)^3 - 4 \cdot (-2)$. Likewise, $-2 \sim 2$, since $(-2)^3 - 4 \cdot (-2) \leq 2^3 - 4 \cdot 2$. But $2 \neq -2$, so $\sim$ is not antisymmetric.

Finally, suppose $x \sim y$ and $y \sim z$. This means that $x^3 - 4x \leq y^3 - 4y$ and $y^3 - 4y \leq z^3 - 4z$. Hence, $x^3 - 4x \leq z^3 - 4z$. Therefore, $x \sim z$, so $\sim$ is transitive. $\square$

Example. Define a relation $\sim$ on $\mathbb{R}^2$ by

$$(a, b) \sim (c, d) \quad \text{means} \quad |ab| \geq |cd|.$$

Check each axiom for a partial order. If the axiom holds, prove it. If the axiom does not hold, give a specific counterexample.

Since $|ab| \geq |ab|$ for all $(a, b) \in \mathbb{R}^2$, it follows that $(a, b) \sim (a, b)$ for all $(a, b) \in \mathbb{R}^2$. Therefore, $\sim$ is reflexive.

$(1, 2) \sim (-1, 2)$, since $|1 \cdot 2| \geq |(-1) \cdot 2|$. Likewise, $(-1, 2) \sim (1, 2)$, since $|(-1) \cdot 2| \geq |1 \cdot 2|$. However, $(1, 2) \neq (-1, 2)$. Therefore, $\sim$ is not antisymmetric.

Finally, suppose $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. Then $|ab| \geq |cd|$ and $|cd| \geq |ef|$. Hence, $|ab| \geq |ef|$. Therefore, $(a, b) \sim (e, f)$. Hence, $\sim$ is transitive. $\square$

Definition. A relation $\sim$ on a set X is a **total order** if:

(a) (Trichotomy) For all $x, y \in X$, exactly one of the following holds: $x \sim y$, $y \sim x$, or $x = y$.

(b) (Transitivity) For all $x, y, z \in X$, if $x \sim y$ and $y \sim z$, then $x \sim z$.

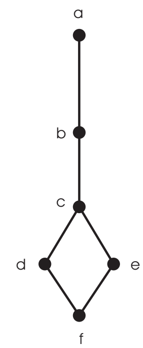
The usual less than relation $<$ is a total order on $\mathbb{Z}$, on $\mathbb{Q}$, and on $\mathbb{R}$. Likewise, you can use the total order relation on $\mathbb{Z}$ to define a lexicographic order on $\mathbb{Z} \times \mathbb{Z}$ which is a total order. Specifically, define a total order $\sim$ on $\mathbb{Z} \times \mathbb{Z}$ as follows: $(x_1, y_1) \sim (x_2, y_2)$ means that

(a) $x_1 < x_2$, or

(b) $x_1 = x_2$ and $y_1 < y_2$.

You can check that the axioms for a total order hold.

Example. Consider the relation defined by the graph below:



Thus, $x < y$ means that $x \neq y$, and there is an upward path of segments from x to y.

Is this relation a total order? You can check cases, using the picture, that the relation is transitive. (This amounts to saying that if there's an upward path from x to y and one from y to z, then there's such a path from x to z. In fact, if you define a relation using a graph in this way, the relation will be transitive.)

However, this graph does not define a total order. Trichotomy fails for d and e, since $d < e$, $e < d$, and $d = e$ are *all* false. $\square$

Definition. Let S be a partially ordered set, and let T be a subset of S.

(a) $s \in S$ is an **upper bound** for T if $s \geq t$ for all $t \in T$.

(b) $s \in S$ is a **lower bound** for T if $s \leq t$ for all $t \in T$.

Thus, an upper bound for a subset is an element which is greater than or equal to everything in the subset; a lower bound for a subset is an element which is less than or equal to everything in the subset. Note that unlike the **largest element** or **smallest element** of a subset, upper and lower bounds don't need to belong to the subset.

For instance, consider the subset $T = (0, 1]$ of $\mathbb{R}$. 2 is an upper bound for T, since $2 \geq x$ for all $x \in T$. 1 is also an upper bound for T. Note that 2 is not an element of T while 1 is an element of T. In fact, any real number greater than or equal to 1 is an upper bound for T.

Likewise, any real number less than or equal to 0 is a lower bound for T.

T has a largest element, namely 1. It does not have a smallest element; the obvious candidate 0 is not in T.

This example shows that a subset may have many — even infinitely many — upper or lower bounds. Among all the upper bounds for a set, there may be one which is *smallest*.

Definition. Let S be a partially ordered set, and let T be a subset of S. An element $s_0 \in S$ is a **least upper bound** for T if:

(a) s_0 is an upper bound for T.

(b) If s is an upper bound for T, then $s_0 \leq s$.

The idea is that s_0 is an upper bound by (a); it's the **least** upper bound, since (b) says s_0 is smaller than any other upper bound.

Definition. Let S be a partially ordered set, and let T be a subset of S. An element $s_0 \in S$ is a **greatest lower bound** for T if:

(a) s_0 is an lower bound for T.

(b) If s is an lower bound for T, then $s_0 \geq s$.

The concepts of least upper bound and greatest lower bound come up often in analysis. I'll give a simple example.

Example. Determine the least upper bound and greatest lower bound for the following sets (if they exist):

(a) The subset $S = (0, 1]$ of $\mathbb{R}$.
(b) The subset $T = (0, +\infty)$ of $\mathbb{R}$. (Thus, T is the positive real axis, not including 0.)

(a) Any real number greater than or equal to 1 is an upper bound for T. Among the upper bounds for S, it's clear that 1 is the *smallest*, so 1 is the *least upper bound* for S.

Likewise, any real number less than or equal to 0 is a lower bound for S. But among the lower bounds for S, it's clear that 0 is the *largest*, so 0 is the *greatest lower bound* for S.

Notice that $1 \in S$, but $0 \notin S$. The least upper bound and greatest lower bound may be contained, or not contained, in the set. $\square$

(b) T has no least upper bound in $\mathbb{R}$; in fact, T has no upper bound in $\mathbb{R}$.

0 is the greatest lower bound for T in $\mathbb{R}$. $\square$
