Set Algebra

Mathematicians tend to prove results about sets as they need them, rather than memorizing and using a large collection of rules. There are a lot of rules involving sets; you'll probably become familiar with the most important ones simply by using them a lot.

Usually you can check informally (for instance, by using a Venn diagram) whether a rule is correct; if necessary, you should be able to write a proof. In most cases, you can give a proof by going back to the definitions of set contructions in terms of elements.

Once you've compiled a collection of known facts about sets, you can use those facts to prove other facts.

There are also various styles for these proofs. You can write a proof formally, as a series of implications or double implications.

Alternatively, you can give a proof that relies more on words.

Example. (**Distributivity**) Let A, B, and C be sets. Prove that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

If X and Y are sets, $X = Y$ if and only if for all x, $x \in X$ if and only if $x \in Y$.

First, I'll give a formal proof, written as a series of double implications:

$$\begin{array}{rcll} x \in A \cap (B \cup C) & \leftrightarrow & x \in A \wedge x \in (B \cup C) & \text{Definition of } \cap \\ & \leftrightarrow & x \in A \wedge (x \in B \vee x \in C) & \text{Definition of } \cup \\ & \leftrightarrow & (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C) & \text{Distributivity of } \wedge \text{ over } \vee \\ & \leftrightarrow & (x \in A \cap B) \vee (x \in A \cap C) & \text{Definition of } \cap \\ & \leftrightarrow & x \in (A \cap B) \cup (A \cap C) & \text{Definition of } \cup \end{array}$$

I've shown that

$$x \in A \cap (B \cup C) \leftrightarrow x \in (A \cap B) \cup (A \cap C).$$

By definition of set equality, this proves that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. $\square$

The idea of the proof was to reduce everything to statements about elements. Then I used logical rules to manipulate the element statements.

Here's an alternative proof written with more words. I'll prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ by showing that each set is contained in the other.

First, I'll show that $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$. Let $x \in A \cap (B \cup C)$. By definition of intersection, this means that $x \in A$ and $x \in B \cup C$.

Now $x \in B \cup C$ means, by definition of union, $x \in B$ or $x \in C$. Combining this with the fact that $x \in A$, this means that either $x \in A$ and $x \in B$, or $x \in A$ and $x \in C$.

By definition of intersection (twice), this means that either $x \in A \cap B$ or $x \in A \cap C$. And by the definition of union, this means that $x \in (A \cap B) \cup (A \cap C)$.

I've shown that if $x \in A \cap (B \cup C)$, then $x \in (A \cap B) \cup (A \cap C)$. By definition of subset, $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$.

Next, I'll show that $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$. Let $x \in (A \cap B) \cup (A \cap C)$. By definition of union, $x \in A \cap B$ or $x \in A \cap C$.

In the first case, $x \in A \cap B$. By definition of intersection, this means $x \in A$ and $x \in B$. Now by constructing a disjunction, $x \in B$ gives $x \in B$ or $x \in C$, and by definition of union, I get $x \in B \cup C$.

Since I know $x \in A$, the definition of intersection gives $x \in A \cap (B \cup C)$.

In the second case, $x \in A \cap C$. By definition of intersection, this means $x \in A$ and $x \in C$. Now by constructing a disjunction, $x \in C$ gives $x \in B$ or $x \in C$, and by definition of union, I get $x \in B \cup C$.

Since I know $x \in A$, the definition of intersection gives $x \in A \cap (B \cup C)$.

Since in both cases I have $x \in A \cap (B \cup C)$, I have shown that if $x \in (A \cap B) \cup (A \cap C)$, then $x \in A \cap (B \cup C)$. By definition of subset, this means that $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$.

Finally, since I've shown that $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$ are each contained in the other, they must be equal: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. $\square$

You can see that the first proof is shorter, but sometimes shorter proofs require more thinking to understand: The proof is shorter because the reasoning is compressed. The second proof is much longer, but maybe the words make more sense to you.

Note: It's also true that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Example. (**DeMorgan's Law**) Let A and B be sets. Prove that

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \quad \text{and} \quad \overline{A \cap B} = \overline{A} \cup \overline{B}.$$

I'll just prove the first statement; the second is similar. This proof will illustrate how you can work with complements. I'll use the *logical* version of DeMorgan's law to do the proof.

Let x be an arbitrary element of the universe.

$$\begin{array}{rcll} x \in \overline{A \cup B} & \leftrightarrow & x \notin A \cup B & \text{Definition of complement} \\ & \leftrightarrow & \neg(x \in A \cup B) & \text{Definition of } \notin \\ & \leftrightarrow & \neg(x \in A \vee x \in B) & \text{Definition of } \cup \\ & \leftrightarrow & \neg(x \in A) \wedge \neg(x \in B) & \text{DeMorgan's law} \\ & \leftrightarrow & (x \notin A) \wedge (x \notin B) & \text{Definition of } \notin \\ & \leftrightarrow & (x \in \overline{A}) \wedge (x \in \overline{B}) & \text{Definition of complement} \\ & \leftrightarrow & x \in \overline{A} \cap \overline{B} & \text{Definition of } \cap \end{array}$$

Therefore, $\overline{A \cup B} = \overline{A} \cap \overline{B}$. $\square$

Example. Let A and B be sets. Prove that $A \cap B \subset A$.

This example will show how you prove a *subset relationship*.

By definition, if X and Y are sets, $X \subset Y$ if and only if for all x, if $x \in X$, then $x \in Y$.

Take an arbitrary element x. Suppose $x \in A \cap B$ (conditional proof). I want to show that $x \in A$.

$x \in A \cap B$ means that $x \in A$ and $x \in B$, by definition of intersection. But $x \in A$ and $x \in B$ implies $x \in A$ (decomposing a conjunction), and this is what I wanted to show. Therefore, $A \cap B \subset A$.

By the way, you usually don't write the logic out in such gory detail. The proof above could be shortened to the following.

$x \in A \cap B$ means that $x \in A$ and $x \in B$, so in particular $x \in A$. Therefore, $A \cap B \subset A$.

The "in particular" substitutes for decomposing the conjunction. $\square$

The procedure I've followed is so common that it's worth pointing it out: To prove a subset relationship (an **inclusion**) $X \subset Y$, take an *arbitrary* element of X and prove that it must be in Y.

In the next example, I'll need the following facts from logic. First, $P \vee \neg P$ is a tautology:

P	$\neg P$	$P \vee \neg P$
T	F	T
F	T	T

Also, $P \wedge (\text{a tautology}) \leftrightarrow P$:

P	a tautology	$P \wedge (\text{a tautology})$
T	T	T
F	T	F

In effect, this means that I can drop tautologies from "and" statements. I'll just call this "Dropping tautologies" in the proof below.

Example. Prove that $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

$$\begin{array}{cc}
x \in (A - B) \cup (B - A) \leftrightarrow & \\
x \in (A - B) \vee x \in (B - A) \leftrightarrow & \text{Definition of union} \\
(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A) \leftrightarrow & \text{Definition of complement} \\
[x \in A \vee (x \in B \wedge x \notin A)] \wedge [x \notin B \vee (x \in B \wedge x \notin A)] \leftrightarrow & \text{Distributivity} \\
(x \in A \vee x \in B) \wedge (x \in A \vee x \notin A)] \wedge (x \notin B \vee x \in B) \wedge (x \notin B \vee x \notin A) \leftrightarrow & \text{Distributivity} \\
(x \in A \vee x \in B) \wedge (x \notin B \vee x \notin A) \leftrightarrow & \text{Dropping tautologies} \\
(x \in A \vee x \in B) \wedge (\neg x \in B \vee \neg x \in A) \leftrightarrow & \text{Definition of "not in"} \\
(x \in A \vee x \in B) \wedge \neg(x \in B \wedge x \in A) \leftrightarrow & \text{DeMorgan} \\
(x \in A \cup B) \wedge \neg(x \in A \cap B) \leftrightarrow & \text{Definition of union and intersection} \\
x \in (A \cup B) - (A \cap B) & \text{Definition of complement}
\end{array}$$

Therefore, $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$. $\square$

Example. Let A be a set. Prove that

$$A \cup \emptyset = A \quad \text{and} \quad A \cap \emptyset = \emptyset.$$

This example will show how you can deal with the empty set.

To prove $A \cup \emptyset = A$, let x be an arbitrary element of the universe. First, by definition of $\cup$,

$$x \in A \cup \emptyset \leftrightarrow (x \in A) \vee (x \in \emptyset).$$

I'll show that $[(x \in A) \vee (x \in \emptyset)] \leftrightarrow (x \in A)$. To prove $P \leftrightarrow Q$, I must prove $P \to Q$ and $Q \to P$.

First, if $x \in A$, then $(x \in A) \vee (x \in \emptyset)$ (constructing a disjunction).

Next, suppose $(x \in A) \vee (x \in \emptyset)$. The second statement $x \in \emptyset$ is false for all x, by definition of $\emptyset$. But the $\vee$-statement is true by assumption, so $x \in A$ must be true by disjunctive syllogism. This proves that if $(x \in A) \vee (x \in \emptyset)$, then $x \in A$.

This completes my proof that $[(x \in A) \vee (x \in \emptyset)] \leftrightarrow (x \in A)$. So

$$\begin{array}{rcll}
x \in A \cup \emptyset & \leftrightarrow & (x \in A) \vee (x \in \emptyset) & \text{Definition of } \cup \\
& \leftrightarrow & x \in A & \text{Proved above}
\end{array}$$

Therefore, $A \cup \emptyset = A$.

To prove that $A \cap \emptyset = \emptyset$, I must prove that for all x, $x \in A \cap \emptyset$ if and only if $x \in \emptyset$.

As usual, x be an arbitrary element of the universe. To prove $x \in A \cap \emptyset$ if and only if $x \in \emptyset$, I must prove that the following implications:

$$(x \in A \cap \emptyset) \to x \in \emptyset \quad \text{and} \quad x \in \emptyset \to (x \in A \cap \emptyset)$$

I'll do this by showing that, in each case, the antecedent (i.e. the "if" part of the statement) is *false* — since by basic logic, if P is false, then $P \to Q$ is true.

For the first implication, consider the statement $x \in A \cap \emptyset$. By definition of intersection,

$$x \in A \cap \emptyset \leftrightarrow (x \in A \wedge x \in \emptyset).$$

Now $x \in \emptyset$ is *false*, by definition of the empty set. Therefore, the conjunction $x \in A \wedge x \in \emptyset$ is also false. Hence, $x \in A \cap \emptyset$ is false.

It follows that the implication $x \in A \cap \emptyset \to x \in \emptyset$ is true, because the "if" part is false.

Likewise, the second implication $x \in \emptyset \to (x \in A \cap \emptyset)$ is true because $x \in \emptyset$ is false, by definition of the empty set.

Since both implications are true, $x \in A \cap \emptyset$ if and only if $x \in \emptyset$. And this in turn proves that $A \cap \emptyset = \emptyset$.

$\square$
