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# Arithmetic Functions

In this section, I'll derive some formulas for $\phi(n)$. I'll also show that $\phi$ has an important property called **multiplicativity**. To put this in the proper context, I'll discuss arithmetic functions, Dirichlet products, and the Möbius inversion formula.

In case you prefer a more direct approach to the formulas and properties of $\phi$, I give an alternative proof of the multiplicativity of $\phi$ in the appendix to this section.

**Definition.** An **arithmetic function** is a function defined on the positive integers which takes values in the real or complex numbers.

For instance, define $f : \mathbb{Z}^+ \to \mathbb{R}$ by $f(n) = \sin n$. Then $f$ is an arithmetic function.

Many functions which are important in number theory are arithmetic functions. For example:

(a) The Euler phi function $\phi$ is an arithmetic function.

(b) Define the **number of divisors function** $\tau : \mathbb{Z}^+ \to \mathbb{Z}^+$ by

$$\tau(n) = (\text{the number of positive divisors of } n).$$

For example, $\tau(12) = 6$, since there are 6 positive divisors of 12 — 1, 2, 3, 4, 6, and 12. $\tau$ is an arithmetic function.

(c) Define the **sum of divisors function** $\sigma : \mathbb{Z}^+ \to \mathbb{Z}^+$ by

$$\sigma(n) = (\text{the sum of the positive divisors of } n).$$

Since 1, 2, 3, 6, 9, and 18 are the positive divisors of 18,

$$\sigma(18) = 1 + 2 + 3 + 6 + 9 + 18 = 39.$$

$\sigma$ is an arithmetic function.

In order to find ways of computing $\phi(n)$, $\tau(n)$, and $\sigma(n)$, we can use the following approach: First, compute the function for $p$, where $p$ is prime.

Next, compute the function for $p^m$, where $p$ is prime and $m \geq 1$.

Finally, for a general number $n$, factor $n$ into a product of powers of primes and use the result for $p^m$.

In order to make the jump from prime powers to an arbitrary integer, we'll show that the functions in question are **multiplicative**. While it's possible to do this directly for each function, we can also prove results which will allow us to use the same approach for $\phi$, $\tau$, and $\sigma$. These results are important for other applications.

**Definition.** The **Möbius function** is the arithmetic function defined by $\mu(1) = 1$, and for $n > 1$,

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n = p_i \cdots p_k, \ p_i \text{ distinct primes} \\ 0 & \text{otherwise} \end{cases}.$$

Thus, $\mu(n) = 0$ if $n$ is divisible by a square.

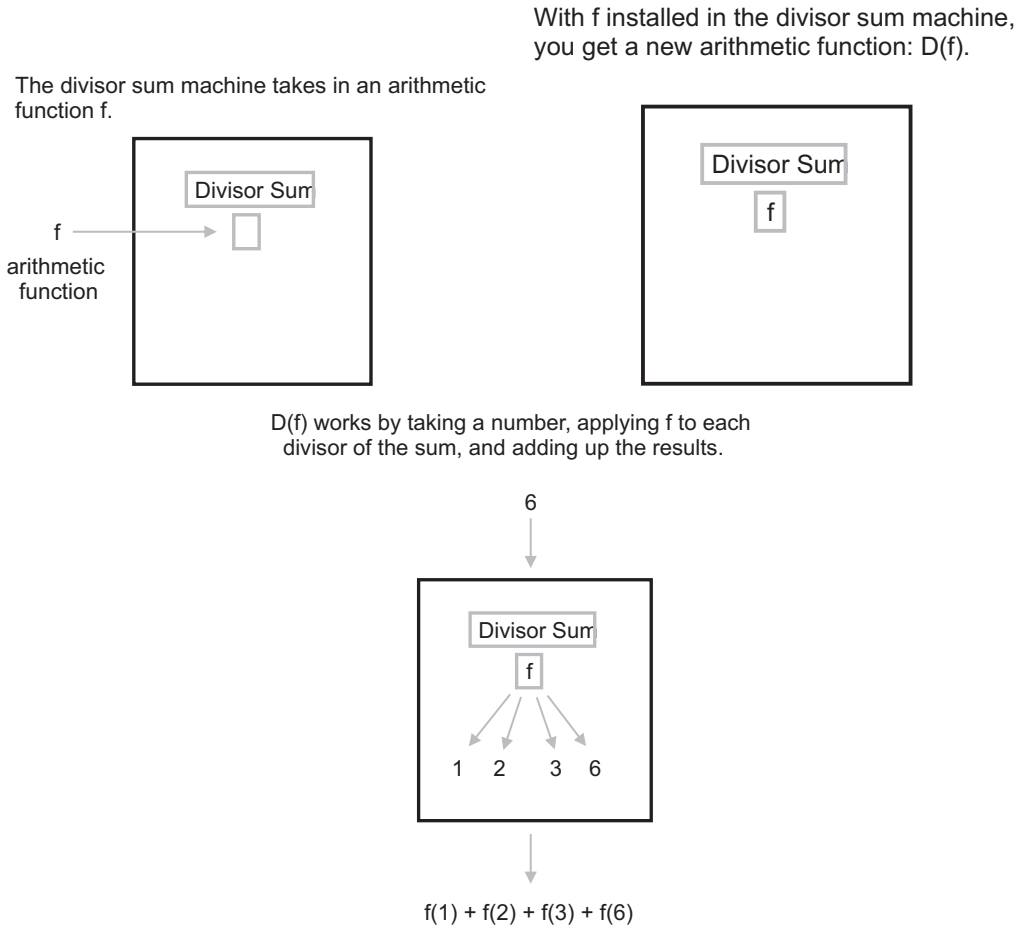
For example, $\mu(6) = 1$, since $6 = 2 \cdot 3$. Likewise, $\mu(30) = -1$, since $30 = 2 \cdot 3 \cdot 5$. But $\mu(12) = 0$ and $\mu(250 = 0$.

**Definition.** If $f$ is an arithmetic function, the **divisor sum** of $f$ is

$$[D(f)](n) = \sum_{d|n} f(d).$$

To save writing, I'll make the convention that when I write "$\sum_{d|n}$", I mean to sum over all the *positive* divisors of a positive integer $n$. Thus, the divisor sum of $f$ evaluated at a positive integer $n$ takes the positive divisors of $n$, plugs them into $f$, and adds up the results. A similar convention will hold for products.

Notice that the divisor sum is a function *which takes an arithmetic function as input and produces an arithmetic function as output.*




**Example.** Suppose $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ is defined by $f(n) = n^2$. Compute $[D(f)](12)$.

$[D(f)](n)$ is the sum of the squares of the divisors of $n$:

$$[D(f)](n) = \sum_{d|n} d^2.$$

thus,

$$[D(f)](12) = \sum_{d|12} d^2 = 1^2 + 2^2 + 3^2 + 4^2 + 6^2 + 12^2 = 210. \quad \square$$

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**Lemma.**

$$[D(\mu)](n) = \sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

**Proof.** The formula for $n = 1$ is obvious.

Suppose $n > 1$. Factor $n$ into a product of powers of primes:

$$n = p_1^{r_1} \cdots p_k^{r_k}.$$

What are the nonzero terms in the sum $\sum_{d|n} \mu(d)$? They will come from $d$'s which are products of single powers of $p_1, \ldots p_k$, and also $d = 1$.

For example, $\mu(p_1 p_2 p_7)$ and $\mu(p_2 p_4)$ would give rise to nonzero terms in the sum, but $\mu(p_3^3 p_8) = 0$.

So

$$\sum_{d|n} \mu(d) = 1 + (\mu(p_1) + \cdots + \mu(p_k)) + (\mu(p_1 p_2) + \mu(p_1 p_3) + \cdots + \mu(p_{k-1} p_k)) + \cdots + \mu(p_1 p_2 \cdots p_k) =$$

$$1 + \binom{k}{1}(-1) + \binom{k}{2}(-1)^2 + \cdots + \binom{k}{k}(-1)^k = (1-1)^k = 0. \quad \square$$

**Example.** Verify the previous lemma for $n = 24$.

The divisor sum is

$$\sum_{d|24} \mu(d) = \mu(1) + \mu(2) + \mu(3) + \mu(4) + \mu(6) + \mu(12) + \mu(24) = 1 + (-1) + (-1) + 0 + 1 + 0 + 0 = 0. \quad \square$$

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**Definition.** If $f$ and $g$ are arithmetic functions, their **Dirichlet product** is

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right).$$

For example for arithmetic functions $f$ and $g$, the Dirichlet product evaluated at 12 is

$$(f * g)(12) = f(1)g(12) + f(2)g(6) + f(3)g(4) + f(4)g(3) + f(6)g(2) + f(12)g(1).$$

**Definition.** Define arithmetic functions

$$I(n) = 1 \quad \text{for all} \quad n \in \mathbb{Z}^+,$$

$$e(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{for all} \quad n \in \mathbb{Z}^+.$$

**Proposition.** Let $f$, $g$, and $h$ be arithmetic functions.

(a) $f * g = g * f$.

(b) $(f * g) * h = f * (g * h)$.

(c) $f * e = f = e * f$.

(d) $f * I = Df = I * f$.

(e) $\mu * I = e$.

**Proof.** For (a), note that divisors of $n$ come in pairs $\left\{d, \frac{n}{d}\right\}$, and that if $\left\{d, \frac{n}{d}\right\}$ is a divisor pair, so is $\left\{\frac{n}{d}, d\right\}$. This means that the same terms occur in both

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right) \quad \text{and} \quad (g * f)(n) = \sum_{d|n} g(d) f\left(\frac{n}{d}\right).$$

Hence, they're equal.

Associativity is a little tedious, so I'll just note that $[(f*g)*h](n)$ and $[f*(g*h)](n)$ are equal to

$$\sum_{\{d,e,f\}} f(d)g(e)h(f).$$

Here the sum runs over all triples of positive numbers $d$, $e$, $f$ such that $def = n$. You can fill in the details.

For (c), note that

$$(f*e)(n) = \sum_{d|n} f(d)e\left(\frac{n}{d}\right) = f(n)e(1) = f(n).$$

($e\left(\frac{n}{d}\right)$ is 0 except when $\frac{n}{d} = 1$, i.e. when $d = n$.)

For (d),

$$(f*I)(n) = \sum_{d|n} f(d)I\left(\frac{n}{d}\right) = \sum_{d|n} f(d)\cdot 1 = \sum_{d|n} f(d) = (Df)(n).$$

For (e), start with $n = 1$:

$$(\mu * I)(1) = \mu(1)I(1) = 1\cdot 1 = 1 = e(1).$$

Now suppose $n > 1$. Then by (d), $\mu * I = D\mu$, so

$$(\mu * I)(n) = (D\mu)(n) = 0 = e(n).$$

Therefore, the formula holds for all $n$. □

The next result is very powerful, but the proof will look easy with all the machinery I've collected.

**Theorem.** (**Möbius Inversion Formula**) If $f$ is an arithmetic function, then

$$f = \mu * Df.$$

**Proof.**

$$\mu * Df = \mu * I * f = e * f = f. \quad \square$$

Next, I'll compute the divisor sum of the Euler phi function.

**Lemma.**

$$[D(\phi)](n) = \sum_{d|n} \phi(d) = n.$$

**Proof.** Let $n$ be a positive integer. Construct the fractions

$$\frac{1}{n}, \quad \frac{2}{n}, \ldots, \frac{n-1}{n}, \quad \frac{n}{n}.$$

Reduce them all to lowest terms. Consider a typical lowest-term fraction $\frac{a}{d}$. Here $d \mid n$ (because it came from a fraction whose denominator was $n$, $a < d$ (because the original fraction was less than 1), and $(a,d) = 1$ (because it's in lowest terms).

Notice that (going the other way) if $\frac{a}{d}$ is a fraction with positive top and bottom which satisfies $d \mid n$, $a < d$, and $(a,d) = 1$, then it *is* one of the lowest-terms fractions. For $dk = n$ for some $k$, and then $\frac{a}{d} = \frac{ka}{kd} = \frac{ka}{n}$ — and the last fraction is one of the original fractions.

How many of the lowest-terms fractions have "$d$" on the bottom? Since the "$a$" on top is a positive number relatively prime to $d$, there are $\phi(d)$ such fractions. Summing over all $d$'s which divide $n$ gives $\sum_{d|n} \phi(d)$. But since every lowest-terms fraction has *some such* "$d$" on the bottom, this sum accounts for all the fractions — and there are $n$ of them. Therefore, $\sum_{d|n} \phi(d) = n$. □

For example, suppose $n = 6$. Then

$$[D(\phi)](6) = \sum_{d|6} \phi(d) = \phi(1) + \phi(2) + \phi(3) + \phi(6) = 1 + 1 + 2 + 2 = 6.$$

**Lemma.** Let $n \geq 1$.

$$\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d}.$$

**Proof.** By Möbius inversion and the previous result,

$$\phi(n) = (\mu * D\phi)(n) = \sum_{d|n} \mu(d) D\phi\left(\frac{n}{d}\right) = \sum_{d|n} \mu(d) \frac{n}{d}. \quad \square$$

For instance, suppose $n = 6$, so $\phi(6) = 2$. Then

$$\sum_{d|6} \mu(d) \frac{6}{d} = \mu(1) \cdot \frac{6}{1} + \mu(2) \cdot \frac{6}{2} + \mu(3) \cdot \frac{6}{3} + \mu(6) \cdot \frac{6}{6} =$$

$$(1)(6) + (-1)(3) + (-1)(2) + (1)(1) = 2.$$

**Theorem.** Let $n \geq 1$.

$$\phi(n) = n \cdot \prod_{\substack{p|n \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right).$$

(By convention, the empty product — the product with no terms — equals 1.)

**Proof.** If $n = 1$, the result is immediate by convention.

If $n > 1$, let $p_1, \ldots, p_k$ be the distinct prime factors of $n$. Then

$$\prod_{\substack{p|n \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right) = \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right) =$$

$$1 - \sum_i \frac{1}{p_i} + \sum_{i \neq j} \frac{1}{p_i p_j} - \cdots + (-1)^k \frac{1}{p_1 p_2 \cdots p_k}.$$

Each term is $\pm\frac{1}{d}$, where $d$ is 1 (the first term) or a product of *distinct* primes. The $(-1)^i$ in front of each term alternates signs according to the number of $p$'s — which is exactly what the Möbius function does. So the expression above is

$$\sum_{d|n} \frac{\mu(d)}{d}.$$

(I can run the sum over *all* divisors, because $\mu(d) = 0$ if $d$ has a repeated prime factor.) Now simply multiply by $n$:

$$n \cdot \prod_{\substack{p|n \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right) = \sum_{d|n} \mu(d) \frac{n}{d} = \phi(n). \quad \square$$

The formula in the theorem is useful for hand-computations of $\phi(n)$. For example, $40 = 2^3 \cdot 5$, so

$$\phi(40) = 40 \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{5}\right) = 16.$$

Likewise, $81 = 3^4$, so

$$\phi(81) = 81 \cdot \left(1 - \frac{1}{3}\right) = 54.$$

**Definition.** An arithmetic function $f$ is **multiplicative** if $(m, n) = 1$ implies

$$f(mn) = f(m)f(n).$$

**Proposition.** $\phi$ is multiplicative — that is, if $(m, n) = 1$, then

$$\phi(mn) = \phi(m)\phi(n).$$

**Proof.** Suppose $(m, n) = 1$. Now

$$\phi(m) = m \cdot \prod_{\substack{p|m \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right) \quad \text{and} \quad \phi(n) = n \cdot \prod_{\substack{q|n \\ q \text{ prime}}} \left(1 - \frac{1}{q}\right).$$

So

$$\frac{\phi(m)\phi(n)}{mn} = \left( \prod_{\substack{p|m \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right) \right) \left( \prod_{\substack{q|n \\ q \text{ prime}}} \left(1 - \frac{1}{q}\right) \right).$$

Since $(m, n) = 1$, the two products have no primes in common. Moreover, the primes that appear in *either* of the products are exactly the prime factors of $mn$. So

$$\frac{\phi(m)\phi(n)}{mn} = \prod_{\substack{r|mn \\ r \text{ prime}}} \left(1 - \frac{1}{r}\right).$$

Hence,

$$\phi(m)\phi(n) = (mn) \cdot \prod_{\substack{r|mn \\ r \text{ prime}}} \left(1 - \frac{1}{r}\right) = \phi(mn). \quad \square$$

**Corollary.** Let $n > 1$, and consider its prime factorization:

$$n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}.$$

(Here the $p$'s are distinct primes, and the $r$'s are positive integers.) Then

$$\phi(n) = (p_1^{r_1} - p_1^{r_1 - 1}) (p_2^{r_2} - p_2^{r_2 - 1}) \cdots (p_k^{r_k} - p_k^{r_k - 1}).$$

Equivalently,

$$\phi(n) = p_1^{r_1-1}(p_1-1)\,p_2^{r_2-1}(p_2-1)\cdots p_k^{r_k-1}(p_k-1).$$

**Proof.** Note that if $i \neq j$, then $(p_i^{r_i}, p_j^{r_j}) = 1$. That is, the prime powers in the prime factorization of $n$ are relatively prime. Recall that if $p$ is prime, then

$$\phi(p^r) = p^r - p^{r-1}.$$

These observations combined with the fact that $\phi$ is multiplicative give

$$\begin{aligned}\phi(n) &= \phi(p_1^{r_1}\,p_2^{r_2}\cdots p_k^{r_k}) \\ &= \phi(p_1^{r_1})\phi(p_2^{r_2})\cdots\phi(p_k^{r_k})) \\ &= (p_1^{r_1} - p_1^{r_1-1})(p_2^{r_2} - p_2^{r_2-1})\cdots(p_k^{r_k} - p_k^{r_k-1})\end{aligned}$$

The second formula follows from the first by factoring out common factors from each $p^r - p^{r-1}$ term. □

Note that in the second formula a prime power $p^r$ gives a "real" $p^{r-1}$ term only if $r > 1$. If $p^1$ is the highest power of $p$ which divides $n$, then the corresponding term in the second formula for $\phi(n)$ is just $p-1$ — and so, if you don't know the value of $r$ in $p^r$, the only term you can assume will appear in $\phi(n)$ is $p-1$.

While the formula in the earlier theorem provided a way of computing $\phi(n)$, the formula in the corollary is often useful in proving results about $\phi(n)$.

**Example.** If $n \geq 3$, then $\phi(n)$ is even. In fact, if $n$ has $k$ odd prime factors, then $2^k \mid \phi(n)$.

To see this, observe first that

$$\phi(2^k) = 2^k - 2^{k-1}.$$

This is even if $2^k \geq 4$.

So suppose that $n$ has $k$ odd prime factors. Each odd prime power factor $p^r$ in the prime factorization of $n$ gives a term $p^r - p^{r-1}$ in the product for $\phi(n)$ in the corollary, and each term $p^r - p^{r-1}$ is even (since it's a difference of odd numbers).

Hence, $\phi(n)$ is divisible by $2^k$.

For example, consider $7623 = 3^2 \cdot 7 \cdot 11^2$. There are 3 odd prime factors, so $\phi(7623)$ should be divisible by 8. And in fact, $\phi(7623) = 3960 = 8 \cdot 495$. □

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**Example.** Find all positive integers $n$ such that $\phi(n) = 8$.

I'll do this in steps. First, I'll show that no prime $\geq 7$ can divide $n$.

At that point, with $n = 2^a \cdot 3^b \cdot 5^c$, I'll get bounds on $a$, $b$, and $c$. That will leave me with 16 cases, which I can check directly.

**Step 1.** No prime greater than 7 divides $n$.

Suppose $p > 7$ and the prime power $p^r$ occurs in the prime factorization of $n$. The formula in the second corollary tells us that there is at least a term $p-1$ in the product for $\phi(n)$. But since $p > 7$, I know that $p$ is at least 11 (the next larger prime). Thus, $p \geq 11$, and so $p - 1 \geq 10$. Then

$$8 = \phi(n) = (p-1)(\text{other terms}) \geq 10 \cdot (\text{other terms}) \geq 10.$$

This is a contradiction, since the right side is larger than 8. Hence, no prime greater than 7 can divide $n$.

**Step 2.** $7 \not| n$.

If $7 \mid n$, then the formula in the second corollary tells us that there is at least a term $7 - 1 = 6$ in the product for $\phi(n)$. So I have

$$8 = \phi(n) = 6 \cdot (\text{other terms}).$$

This is a contradiction, since $6 \nmid 8$.

At this point, I know that $n = 2^a\, 3^b\, 5^c$.

**Step 3.** $a \leq 4$ and $b \leq 1$ and $c \leq 1$.

I have

$$8 = \phi(n) = 2^{a-1}(2-1)3^{b-1}(3-1)5^{c-1}(5-1).$$

Suppose $a \geq 5$. The factor $2^{a-1}$ is at least $2^4 = 16$, but $16 > 8$. Hence, $a \leq 4$.

Suppose $b \geq 2$. The factor $3^{b-1}$ is at least $3^1$, so 3 divides the right side. But $3 \nmid 8$. Hence, $b \leq 1$.

Suppose $c \geq 2$. The factor $5^{c-1}$ is at least $5^1$, so 5 divides the right side. But $5 \nmid 8$. Hence, $c \leq 1$.

At this point, I know $a = 0, 1, 2, 3$ and $b = 0, 1$ and $c = 0, 1$. I could probably eliminate some possibilities by additional analysis, but with $4 \cdot 2 \cdot 2 = 16$ cases, I can just check by hand.

**Step 4.** Check the remaining cases by hand.

| $a$ | $b$ | $c$ | $n$ | $\phi(n)$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 |
| 0 | 0 | 1 | 5 | 4 |
| 0 | 1 | 0 | 3 | 2 |
| 0 | 1 | 1 | 15 | 8 |
| 1 | 0 | 0 | 2 | 1 |
| 1 | 0 | 1 | 10 | 4 |
| 1 | 1 | 0 | 6 | 2 |
| 1 | 1 | 1 | 30 | 8 |

| $a$ | $b$ | $c$ | $n$ | $\phi(n)$ |
|---|---|---|---|---|
| 2 | 0 | 0 | 4 | 2 |
| 2 | 0 | 1 | 20 | 8 |
| 2 | 1 | 0 | 12 | 4 |
| 2 | 1 | 1 | 60 | 16 |
| 3 | 0 | 0 | 8 | 4 |
| 3 | 0 | 1 | 40 | 16 |
| 3 | 1 | 0 | 24 | 24 |
| 3 | 1 | 1 | 120 | 32 |

The numbers are 15, 20, 24, 30. □

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**Appendix: An alternate proof of multiplicativity for $\phi(n)$**

In this appendix, I'll give a direct proof of the multiplicativity of $\phi(n)$ which doesn't use any results on arithmetic functions.

**Theorem.** If $(m, n) = 1$, then

$$\phi(mn) = \phi(m)\phi(n).$$

**Proof.** I list the numbers from 1 to $mn$ and count the number that are relatively prime to $mn$.

$$\begin{array}{ccccccc}
1 & 2 & 3 & 4 & \cdots & m \\
m+1 & m+2 & m+3 & m+4 & \cdots & 2m \\
2m+1 & 2m+2 & 2m+3 & 2m+4 & \cdots & 3m \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
am+1 & am+2 & am+3 & am+4 & \cdots & am+m \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
(n-1)m+1 & (n-1)m+2 & (n-1)m+3 & (n-1)m+4 & \cdots & (n-1)m+m
\end{array}$$

A typical column looks like:

$$\begin{array}{c} k \\ m+k \\ 2m+k \\ \vdots \\ am+k \\ \vdots \\ (n-1)m+k \end{array}$$

Note that $(m,k)$ divides all the numbers in this column. Moreover, $(m,k) \mid m \mid mn$. Thus, if $(m,k) \neq 1$, then $(m,k)$ is a nontrivial divisor of each number in this column and of $mn$.

Hence, if $(m,k) \neq 1$, all the numbers in this column are *not* relatively prime to $mn$.

Therefore, since I'm counting numbers that are relatively prime to $mn$, I need only consider columns where $(m,k) = 1$. There are $\phi(m)$ such columns.

So consider a column where $(m,k) = 1$. It contains the numbers

$$\{k,\ m+k,\ 2m+k, \ldots (n-1)m+k\}.$$

Start with $\{0, 1, 2, \ldots n-1\}$, the standard residue system mod $n$. Since $(m,n) = 1$, multiplying by $m$ produces another complete residue system:

$$\{0,\ m,\ 2m, \ldots (n-1)m\}.$$

Adding $k$ must also give a complete residue system:

$$\{k,\ m+k,\ 2m+k, \ldots (n-1)m+k\}.$$

This is the column in question, so I've shown that such a column is a complete residue system mod $n$. It follows that $\phi(n)$ of these numbers are relatively prime to $n$.

Thus, I have $\phi(m)$ columns, all of whose elements are relatively prime to $m$, and in each such column $\phi(n)$ of the elements are relatively prime to $n$. Thus, there are $\phi(m)\phi(n)$ numbers in $\{1, 2, \ldots mn\}$ which are relatively prime to $mn$. □

Let's look at how the proof works with a specific example. Take $m = 9$ and $n = 4$. List the numbers from 1 to 36:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
| 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 |
| 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 |

You can see that the columns beginning with 3, 6, and 9 — the numbers $k$ with $(9,k) \neq 1$ — contain only numbers that are not relatively prime to 9 (and hence, are not relatively prime to 36). Removing these columns, I have $\phi(9) = 6$ columns left: The ones beginning with 1, 2, 4, 5, 7, and 8.

Pick any one of those columns — say the one beginning with 7, which contains $\{7, 16, 25, 34\}$. Note that these numbers reduce mod 4 to $\{3, 0, 1, 2\}$, which is a complete residue system mod 4. And exactly $\phi(4) = 2$ of these numbers — in this case, 7 and 25 — are relatively prime to 4.

Thus, there are $\phi(9)\phi(4) = 6 \cdot 2 = 12$ numbers in $\{1, 2, \ldots 36\}$ which are relatively prime to $9 \cdot 4 = 36$, and so $\phi(9 \cdot 4) = \phi(9)\phi(4)$.