Fibonacci Numbers

The **Fibonacci numbers** are defined by the following recursive formula:

$$f_0 = 1, \quad f_1 = 1,$$

$$f_n = f_{n-1} + f_{n-2} \quad \text{for} \quad n \geq 2.$$

Thus, each number in the sequence (after the first two) is the sum of the previous two numbers.

(Some people start numbering the terms at 1, so $f_1 = 1$, $f_2 = 1$, and so on. But the recursion is the same.)

The first few Fibonacci numbers are:

$$1, \quad 1, \quad 2, \quad 3, \quad 5, \quad 8, \ldots.$$

Fibonacci numbers have been extensively studied. Koshy [1] and Rao [2] have extensive lists of Fibonacci identities; Koshy also has many applications. The *Fibonacci Quarterly* is a journal devoted to Fibonacci numbers and related topics.

Example. Express each of the following as a single Fibonacci number.

(a) $f_{5n+1} + f_{5n+2}$.

(b) $f_{349} - f_{348}$.

(c) $f_{2n+7} + f_{2n+4} + f_{2n+5}$.

(a) The number after $5n + 1$ and $5n + 2$ is $5n + 3$, so

$$f_{5n+1} + f_{5n+2} = f_{5n+3}. \quad \square$$

(b) Since $f_{347} + f_{348} = f_{349}$,

$$f_{349} - f_{348} = f_{347}. \quad \square$$

(c)

$$f_{2n+7} + f_{2n+4} + f_{2n+5} = f_{2n+7} + f_{2n+6} = f_{2n+8}. \quad \square$$

Example. Prove that if $n \geq -1$, then

$$f_{n+5} + f_{n+1} = 3f_{n+3}.$$

$$\begin{aligned} f_{n+5} + f_{n+1} &= (f_{n+4} + f_{n+3}) + (f_{n+3} - f_{n+2}) \\ &= (f_{n+4} - f_{n-2}) + 2f_{n+3} \\ &= f_{n+3} + 2f_{n+3} \\ &= 3f_{n+3} \quad \square \end{aligned}$$

Many results about Fibonacci numbers can be proved by induction.

Example. Prove that

$$f_0 + f_1 + \cdots + f_n = f_{n+2} - 1 \quad \text{for} \quad n \geq 0.$$

For $n = 0$, the left side is $f_0 = 1$ and the right side is

$$f_2 - 1 = 2 - 1 = 1.$$

The result is true for $n = 0$.
Suppose the result holds for n:

$$f_0 + f_1 + \cdots + f_n = f_{n+2} - 1.$$

I'll prove it for $n + 1$.

$$\begin{aligned} f_0 + f_1 + \cdots + f_n + f_{n+1} &= (f_{n+2} - 1) + f_{n+1} \\ &= (f_{n+2} + f_{n+1}) - 1 \\ &= f_{n+3} - 1 \end{aligned}$$

This proves the result for $n + 1$, so the result is true for all $n \geq 0$ by induction. □

Example. Prove that for $n \geq 0$,

$$f_n f_{n+2} = f_{n+1}^2 + (-1)^n.$$

For $n = 0$, the left side is

$$f_0 f_2 = 1 \cdot 2 = 2.$$

The right side is

$$f_1^2 + (-1)^0 = 1^2 + 1 = 2.$$

The result is true for $n = 0$.
Assume the result for n:

$$f_n f_{n+2} = f_{n+1}^2 + (-1)^n, \quad \text{so} \quad f_{n+1}^2 = f_n f_{n+2} - (-1)^n.$$

Prove the result for $n + 1$:

$$\begin{aligned} f_{n+1} f_{n+3} &= f_{n+1}(f_{n+1} + f_{n+2}) \\ &= f_{n+1}^2 + f_{n+1} f_{n+2} \\ &= f_n f_{n+2} - (-1)^n + f_{n+1} f_{n+2} \\ &= (f_n + f_{n+1}) f_{n+2} - (-1)^n \\ &= f_{n+2}^2 + (-1)^{n+1} \end{aligned}$$

This proves the result for $n + 1$, so it's true for $n \geq 0$ by induction. □

Example. (**An explicit formula for the Fibonacci numbers**)

(a) Let

$$\alpha = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1 - \sqrt{5}}{2}.$$

Prove:

$$\begin{aligned} \alpha + \beta &= 1 \\ \alpha - \beta &= \sqrt{5} \\ \alpha^2 &= \alpha + 1 \\ \beta^2 &= \beta + 1 \end{aligned}$$

(b) Prove that

$$f_n = \frac{1}{\sqrt{5}}\alpha^{n+1} - \frac{1}{\sqrt{5}}\beta^{n+1} \quad \text{for} \quad n \geq 0.$$

(a)

$$\alpha + \beta = \frac{1+\sqrt{5}}{2} + \frac{1-\sqrt{5}}{2} = \frac{1+\sqrt{5}+1-\sqrt{5}}{2} = \frac{2}{2} = 1.$$

$$\alpha - \beta = \frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} = \frac{1+\sqrt{5}-1+\sqrt{5}}{2} = \frac{2\sqrt{5}}{2} = \sqrt{5}.$$

For the third and fourth equations, note that α and β are roots of the quadratic equation

$$x^2 - x - 1 = 0.$$

So:

$$\alpha^2 - \alpha - 1 = 0 \quad \text{and hence} \quad \alpha^2 = \alpha + 1.$$

$$\beta^2 - \beta - 1 = 0 \quad \text{and hence} \quad \beta^2 = \beta + 1.$$

(b) For $n = 0$, I have $f_0 = 1$. The right side of the equation above becomes

$$\frac{1}{\sqrt{5}}\alpha - \frac{1}{\sqrt{5}}\beta = \frac{1}{\sqrt{5}} \cdot \sqrt{5} = 1.$$

The result is true for $n = 0$.
For $n = 1$, I have $f_1 = 1$. The right side of the equation above becomes

$$\frac{1}{\sqrt{5}}\alpha^2 - \frac{1}{\sqrt{5}}\beta^2 = \frac{1}{\sqrt{5}}(\alpha^2 - \beta^2) = \frac{1}{\sqrt{5}}\left[(\alpha+1) - (\beta+1)\right] =$$

$$\frac{1}{\sqrt{5}}(\alpha - \beta) = \frac{1}{\sqrt{5}} \cdot \sqrt{5} = 1.$$

The result is true for $n = 1$.
Assume that the result is true for all $k \leq n$. In particular,

$$f_n = \frac{1}{\sqrt{5}}\alpha^{n+1} - \frac{1}{\sqrt{5}}\beta^{n+1}.$$

$$f_{n-1} = \frac{1}{\sqrt{5}}\alpha^{n} - \frac{1}{\sqrt{5}}\beta^{n}.$$

I'll prove the result for $n + 1$.

$$f_{n+1} = f_n + f_{n-1} = \left(\frac{1}{\sqrt{5}}\alpha^{n+1} - \frac{1}{\sqrt{5}}\beta^{n+1}\right) + \left(\frac{1}{\sqrt{5}}\alpha^{n} - \frac{1}{\sqrt{5}}\beta^{n}\right) =$$

$$\frac{1}{\sqrt{5}}\left[\left(\alpha^{n+1} + \alpha^n\right) + \left(\beta^{n+1} + \beta^n\right)\right] = \frac{1}{\sqrt{5}}\left[\alpha^n(\alpha+1) + \beta^n(\beta+1)\right] =$$

$$\frac{1}{\sqrt{5}}\left(\alpha^n \cdot \alpha^2 + \beta^n \cdot \beta^2\right) = \frac{1}{\sqrt{5}}\left(\alpha^{n+2} + \beta^{n+2}\right).$$

This proves the result for $n + 1$, so the result is true for all $n \geq 0$ by induction. □

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[1] Thomas Koshy, Fibonacci and Lucas Numbers with Applications. New York: John Wiley and Sons, 2001.

[2] K. Subba Rao, *Some properties of Fibonacci numbers*, American Mathematical Monthly, (10) 60 (1953), 680–684.