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Fractions and Rational Numbers

Definition. A **rational number** is a real number which can be written as $\dfrac{a}{b}$, where a and b are integers and $b \neq 0$. A real number which is not rational is **irrational**.

Example. Prove that if p is prime, then $\sqrt{p}$ is irrational.

To prove this, suppose to the contrary that $\sqrt{p}$ is rational. Write $\sqrt{p} = \dfrac{a}{b}$, where a and b are integers and $b \neq 0$. I may assume that $(a, b) = 1$ — if not, divide out any common factors.

Now

$$b\sqrt{p} = a \quad \text{so} \quad b^2 p = a^2.$$

Since $p \mid a^2$ and p is prime, $p \mid a$. Write $a = pc$. Then

$$b^2 p = p^2 c^2, \quad \text{so} \quad b^2 = pc^2.$$

Now $p \mid b^2$, so $p \mid b$. Thus, p is a common factor of a and b contradicting my assumption that $(a, b) = 1$. It follows that $\sqrt{p}$ is irrational. $\square$

More generally, suppose a_0, ..., a_{n-1} are integers and

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

Then the roots are either integers or irrational.

If b is an integer such that $b > 1$, and a is a positive integer, then for some $n \geq 0$ I can write a uniquely in the form

$$a = \sum_{i=0}^{n} a_i b^i.$$

This is called the **base b expansion of a**.

Note that

$$a = a_n b^n + a_{n-1} b^{n-1} + \cdots + a_1 b + a_0.$$

The notation is $(a_n\, a_{n-1}\, \ldots a_1\, a_0)_b$, with the subscript b denoting the base. We omit the subscript for number given in base-10.

Thus, the value of a is obtained by plugging $x = b$ into the polynomial

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

The standard way to do this by hand is to use **synthetic division**.

Example. Convert $(7513)_8$ to base-10. Use synthetic division:

7	5	1	3
	56	488	3912
7	61	489	3915

Thus, $(7513)_8 = 3915$. $\square$

To convert from base-10 to base-b, we just have to undo the process above. I divide the number by the base, noting the quotient and the remainder. Then I divide the quotient by the base, and so on. The successive remainders give the base-b digits (backwards).

Example. Convert 3915 to base-8. Divide 3915 by 8. The quotient is 489 and the remainder is 3:

$$\begin{array}{cc} 489 & 3915 \\ & 3 \end{array}$$

Divide 489 by 8. The quotient is 61 and the remainder is 1:

$$\begin{array}{ccc} 61 & 489 & 3915 \\ & 1 & 3 \end{array}$$

Divide 61 by 8. The quotient is 7 and the remainder is 5:

$$\begin{array}{cccc} 7 & 61 & 489 & 3915 \\ & 5 & 1 & 3 \end{array}$$

Since 7 is less than 8, I can stop here. The answer is $(7\,5\,1\,3)_8$. □

Note that if you want to convert between base-b and base-c, you could just do

$$(\text{base-}b) \to (\text{base-}10) \to (\text{base-}c)$$

What about a positive number which is not an integer? I can write any positive real number as a sum of a positive integer and a real number between 0 and 1. I already know how to convert positive integers to base-b.

So suppose b is an integer such that $b > 1$, and a is a real number between 0 and 1 (inclusive). Then a can be written uniquely in the form

$$a = \sum_{i=1}^{\infty} a_i \cdot \frac{1}{b^i}.$$

Rather than proving this fact, I'll merely recall the standard algorithm for computing such an expansion: Subtract from a as many $\dfrac{1}{b}$'s as possible, subtract as many $\dfrac{1}{b^2}$'s from what's left, and so on.

Here is a recursive procedure which generates base b expansions:

$$x_0 = a$$

$$a_i = [b \cdot x_{i-1}], \quad x_i = b \cdot x_{i-1} - [b \cdot x_{i-1}] \quad \text{for} \quad i \geq 1.$$

To see why this corresponds to the standard algorithm, note that at the first stage I'm trying to find $k \geq 0$ such that

$$a - \frac{k}{b} \geq 0 \quad \text{and} \quad a - \frac{k+1}{b} < 0.$$

These equations are equivalent to

$$ba - k \geq 0 \quad \text{and} \quad ba - (k+1) < 0.$$

Equivalently,

$$ba \geq k \quad \text{and} \quad ba < k+1.$$

That is, $k = [ba]$, and a corresponds to x_i.

It's convenient to arrange the computations in a table, as shown below.

Example. Find 0.4 in base 7.

I fill in the rows from left to right. Starting with an x, multiply by $b = 7$ to fill in the third column. Take the greatest integer of the result to fill in the a-column of the next row. Subtract the a-value from the last bx-value to get the next x, and continue. You can check that this is the algorithm described above.

a	x	bx
–	0.4	2.8
2	0.8	5.6
5	0.6	4.2
4	0.2	1.4
1	0.4	2.8

The expansion clearly repeats after this, since I'm getting 0.4 for x again. Thus,

$$0.4 = (0.\overline{2541})_7. \quad \square$$

Definition. The decimal expansion $x = .a_1 a_2 \ldots$ **terminates** if there is a number $N > 0$ such that $a_k = 0$ for $k > n$.

In this case,

$$x = \frac{a_1 \cdot 10^{n-1} + a_2 \cdot 10^{n-2} + \cdots + a_n}{10^n}.$$

Hence, x is rational.

In fact, rational numbers in $(0, 1)$ with terminating decimal are exactly the rational numbers of the form $\dfrac{p}{2^a 5^b}$ for $p > 0$ and $a, b \geq 0$.

Suppose a rational number has the form $\dfrac{p}{2^a 5^b}$ for $p > 0$ and $a, b \geq 0$. To see this, multiply the top and bottom by a power of 2 or a power of 5 to get a power of 10 on the bottom. Then $\dfrac{p}{2^a 5^b} = \dfrac{q}{10^c}$, which is represented by a terminating decimal with q being the "decimal part". For example,

$$\frac{17}{40} = \frac{17}{2^3 \cdot 5} = \frac{17 \cdot 5^2}{2^3 \cdot 5^3} = \frac{425}{10^3} = \frac{425}{1000} = 0.425.$$

Going the other way, note that

$$0.a_1\, a_2\, \ldots a_n = \frac{a_1 \cdot 10^n + a_2 \cdot 10^{n-1} + \cdots a_n}{10^n} = \frac{a_1 \cdot 10^n + a_2 \cdot 10^{n-1} + \cdots a_n}{2^n \cdot 5^n}.$$

For instance,

$$0.4173 = \frac{4173}{10000} = \frac{4 \cdot 10^3 + 1 \cdot 10^2 + 7 \cdot 10 + 3}{10^4} = \frac{4 \cdot 10^3 + 1 \cdot 10^2 + 7 \cdot 10 + 3}{2^4 \cdot 5^4}.$$

Thus, a terminating decimal has the form $\dfrac{p}{2^a 5^b}$ for $p > 0$ and $a, b \geq 0$.

A decimal expansion $x = .a_1\, a_2\, \ldots$ is **periodic** with period k if there is a positive integer N such that $a_n = a_{n+k}$ for all $n \geq N$.

Proposition. A periodic decimal expansion represents a rational number.

Proof. (Sketch) First consider the simplest case of a periodic decimal

$$0.a_1\, a_2\, \ldots a_k\, a_1\, a_2\, \ldots a_k\, \ldots .$$

This is a geometric series with first term $a = \dfrac{a_1 \cdot 10^{k-1} + a_2 \cdot 10^{k-2} + \cdots a_k}{10^k}$ and ratio $r = \dfrac{1}{10^k}$. a and r are both rational The sum of such a geometric series is $\dfrac{a}{1-r}$, which is also a rational number.

Suppose there is a **pre-period** — an initial segment before the repeating part:

$$x = 0.b_1\, b_2\, \ldots b_j\, a_1\, a_2\, \ldots a_k\, a_1\, a_2\, \ldots a_k\, \ldots .$$

This is a sum of two rational numbers: The rational number corresponding to the terminating decimal $0.b_1\, b_2 \ldots b_j$ and the rational number corresponding to the periodic part $a_1\, a_2\, \ldots a_k\, a_1\, a_2\, \ldots a_k\, \ldots$, shifted by j places. Explicitly, if $a = \dfrac{a_1 \cdot 10^{k-1} + a_2 \cdot 10^{k-2} + \cdots a_k}{10^k}$ and $r = \dfrac{1}{10^k}$, then

$$x = \frac{b_1 \cdot 10^{j-1} + b_2 \cdot 10^{j-2} + \cdots b_j}{10^j} + \frac{1}{10^j} \cdot \frac{a}{1-r}.$$

Once again, this is rational. □

Example. Express $0.\overline{473}$ as a rational number in lowest terms.

Since the number has period 3, I multiply both sides by 10^3:

$$\begin{aligned} x &= 0.\overline{473} = 0.473473\ldots \\ 1000x &= 473.473473\ldots \end{aligned}$$

Next, subtract the first equation from the second:

$$\begin{aligned} 1000x &= 473.473473\ldots \\ x &= 0.473473\ldots \\ \hline 999x &= 473 \\ x &= \frac{473}{999} \end{aligned}$$

□

Example. Express $(0.24\overline{122})_{10}$ as a rational number in lowest terms.

Since the number has period 3, I multiply both sides by 10^3:

$$\begin{aligned} x &= (0.24\overline{122})_{10} = 0.24122122\ldots \\ 1000x &= 241.22122122\ldots \end{aligned}$$

Next, subtract the first equation from the second:

$$\begin{aligned} 1000x &= 241.22122122\ldots \\ x &= 0.24122122\ldots \\ \hline 999x &= 240.98 \\ x &= \frac{240.98}{999} = \frac{24098}{99900} = \frac{12049}{49950} \end{aligned}$$

□

Example. Express $(0.\overline{473})_8$ as a base-10 rational number in lowest terms.

Since the number has period 3, I multiply both sides by $8^3 = 512$:

$$\begin{aligned} x &= (0.\overline{473})_8 = (0.473473\ldots)_8 \\ 512 \cdot x &= (473.473473\ldots)_8 \end{aligned}$$

Next, subtract the first equation from the second, being careful about the bases: I have base-10 on the left, but base-8on the right.

$$\begin{aligned} 512 \cdot x &= (473.473473\ldots)_8 \\ x &= (0.473473\ldots)_8 \\ \hline 511x &= (473)_8 = 315 \\ x &= \frac{315}{511} = \frac{45}{73} \end{aligned} \quad \square$$

In the next two problems, I'll use the formula for the sum of a geometric series:

$$a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots = \frac{a}{1-r}.$$

Example. Suppose b is an integer and $b > 4$. Express the following as a rational function of b:

$$(0.13\,13\,\overline{13}\ldots)_b.$$

Using the formula for the sum of a geometric series, I have

$$\begin{aligned} (0.13\,13\,\overline{13}\ldots)_b &= \frac{1}{b} + \frac{3}{b^2} + \frac{1}{b^3} + \frac{3}{b^4} + \frac{1}{b^5} + \frac{3}{b^6} + \cdots \\ &= \frac{b+3}{b^2} + \frac{b+3}{b^4} + \frac{b+3}{b^6} + \cdots \\ &= \frac{\dfrac{b+3}{b^2}}{1 - \dfrac{1}{b^2}} \\ &= \frac{b+3}{b^2-1} \end{aligned} \quad \square$$

Example. Suppose b is an integer and $b > 3$. Express the following as a rational function of b:

$$(0.(b-2)1\,(b-2)1\,\overline{(b-2)1}\ldots)_b.$$

Using the formula for the sum of a geometric series, I have

$$\begin{aligned} (0.(b-2)1\,(b-2)1\,\overline{(b-2)1}\ldots)_b &= \frac{b-2}{b} + \frac{1}{b^2} + \frac{b-2}{b^3} + \frac{1}{b^4} + \cdots \\ &= \frac{b^2-2b+1}{b^2} + \frac{b^2-2b+1}{b^4} + \cdots \\ &= \frac{\dfrac{b^2-2b+1}{b^2}}{1 - \dfrac{1}{b^2}} \\ &= \frac{b^2-2b+1}{b^2-1} \\ &= \frac{b-1}{b+1} \end{aligned} \quad \square$$

Proposition. A rational number can be represented by either a terminating decimal, or a periodic decimal.

Proof. (Sketch) Suppose $\dfrac{p}{q}$ is a rational number in lowest terms, so $(p,q) = 1$, and $0 < \dfrac{p}{q} < 1$.

I've already shown that $\dfrac{p}{q}$ ca be represented by a terminating decimal if and only if $q = 2^a 5^b$ for some $a, b \geq 0$.

I'll consider the case where $(q, 10) = 1$, so q is not divisible by 2 or by 5. By Euler's theorem,

$$10^{\phi(q)} = 1 \pmod{q}.$$

Since some positive power of 10 is equal to 1 mod q, there must be a smallest positive power n such that $10^n = 1 \pmod{q}$. (This is called the **order** of 10 mod q.) Thus,

$$10^n = mq + 1 \quad \text{for some} \quad m \in \mathbb{Z}^+.$$

I have

$$10^n \cdot \frac{p}{q} = \frac{(mq+1)p}{q} = \frac{mqp}{q} + \frac{p}{q} = mp + \frac{p}{q}.$$

On the other hand, I have the decimal expansion

$$\frac{p}{q} = \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n} + x.$$

Here x represents the remainder of the decimal expansion, so

$$x = \frac{a_{n+1}}{10^{n+1}} + \frac{a_{n+2}}{10^{n+2}} + \frac{a_{n+3}}{10^{n+3}} + \cdots.$$

Note that

$$10^n x = \frac{a_{n+1}}{10} + \frac{a_{n+2}}{10^2} + \frac{a_{n+3}}{10^3} + \cdots.$$

Hence, $0 < 10^n x < 1$.

So multiplying the equation for $\dfrac{p}{q}$ by 10^n, I get

$$10^n \cdot \frac{p}{q} = \left(10^{n-1} a_1 + 10^{n-2} a_2 + \cdots + a_n\right) + 10^n x.$$

Comparing the two equations for $10^n \cdot \dfrac{p}{q}$, I have

$$mp + \frac{p}{q} = \left(10^{n-1} a_1 + 10^{n-2} a_2 + \cdots + a_n\right) + 10^n x.$$

I have an integer on either side, namely mp and $10^{n-1}a_1 + 10^{n-2}a_2 + \cdots + a_n$. I also have on either side numbers in the range $(0,1)$, namely $\dfrac{p}{q}$ and $10^n x$. This is only possible if $\dfrac{p}{q} = 10^n x$. This means that at the $(n+1)^{\text{st}}$ place the decimal being constructed is the decimal for the original $\dfrac{p}{q}$. Hence, the decimal must repeat after that point.

I'll omit the case where $q = 2^a 5^b q'$, where $(q', 10) = 1$. In this case, the decimal has a pre-preriod before it begins to repeat. □

For example, consider the rational fraction $\dfrac{10}{21}$. I have $\phi(21) = 12$, and checking powers I find that $10^6 = 1 \pmod{21}$, and this is the smallest positive power of 10 equal to 1 mod 21. Thus, I expect the decimal to have period 6. In fact,

$$\frac{10}{21} = 0.809523\,809523\ldots.$$