

Infinite Continued Fractions

If $[a_0; a_1, a_2, \dots]$ is an infinite continued fraction, I want to define its *value* to be the limit of the convergents:

$$\lim_{k \rightarrow \infty} c_k, \quad \text{where } c_k \text{ is the } k\text{-th convergent.}$$

For this to make sense, I need to show that this limit exists.

In what follows, take as given an infinite continued fraction $[a_0; a_1, a_2, \dots]$. Let

$$\begin{aligned} p_0 &= a_0, & q_0 &= 1 \\ p_1 &= a_1 a_0 + 1, & q_1 &= a_1 \\ p_k &= a_k p_{k-1} + p_{k-2}, & q_k &= a_k q_{k-1} + q_{k-2}, & k &\geq 2, \\ c_k &= \frac{p_k}{q_k}. \end{aligned}$$

Proposition.

$$(a) \quad c_k - c_{k-1} = \frac{(-1)^{k-1}}{q_{k-1} q_k}.$$

$$(b) \quad c_k - c_{k-2} = \frac{a_k (-1)^k}{q_{k-2} q_k}.$$

Proof. (a)

$$c_k - c_{k-1} = \frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} = \frac{p_k q_{k-1} - p_{k-1} q_k}{q_{k-1} q_k} = \frac{(-1)^{k-1}}{q_{k-1} q_k}.$$

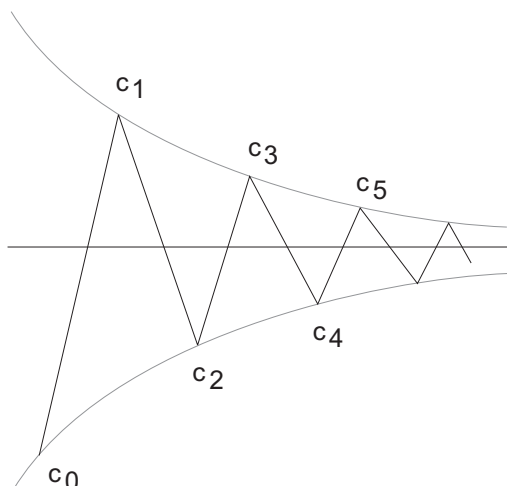
(b)

$$\begin{aligned} c_k - c_{k-2} &= \frac{p_k}{q_k} - \frac{p_{k-2}}{q_{k-2}} = \frac{p_k q_{k-2} - p_{k-2} q_k}{q_{k-2} q_k} = \frac{(a_k p_{k-1} + p_{k-2}) q_{k-2} - p_{k-2} (a_k q_{k-1} + q_{k-2})}{q_{k-2} q_k} = \\ &= \frac{a_k (p_{k-1} q_{k-2} - p_{k-2} q_{k-1})}{q_{k-2} q_k} = \frac{a_k \cdot (-1)^{k-2}}{q_{k-2} q_k} = \frac{a_k (-1)^k}{q_{k-2} q_k}. \quad \square \end{aligned}$$

Proposition.

$$c_1 > c_3 > c_5 > \dots > c_4 > c_2 > c_0.$$

That is, the odd convergents get smaller, the even convergents get bigger, and any odd convergent is bigger than any even convergent.



Likewise, consider the sequence of even convergents:

$$\cdots > c_4 > c_2 > c_0.$$

This is an increasing sequence of numbers that's bounded above — by any odd convergent, for example. The result from analysis mentioned above says that the sequence has a limit:

$$\lim_{k \rightarrow \infty} c_{2k} \text{ exists.}$$

I have to show that the two limits agree.

The previous result implies that $q_{2k} \geq 2k$ and $q_{2k+1} \geq 2k + 1$, so

$$0 \leq c_{2k+1} - c_{2k} = \frac{(-1)^{2k+1-1}}{q_{2k}q_{2k+1}} \leq \frac{1}{(2k)(2k+1)}.$$

Now let $k \rightarrow \infty$. $\frac{1}{(2k)(2k+1)} \rightarrow 0$, so by the Squeezing Theorem of calculus,

$$\begin{aligned} \lim_{k \rightarrow \infty} (c_{2k+1} - c_{2k}) &= 0 \\ \lim_{k \rightarrow \infty} c_{2k+1} &= \lim_{k \rightarrow \infty} c_{2k} \end{aligned}$$

Since the odd and even terms approach the same limit, $\lim_{k \rightarrow \infty} c_k$ exists. \square

Definition. $[a_0; a_1, a_2, \dots] = \lim_{k \rightarrow \infty} c_k$.

What can I say about the value of an infinite continued fraction?

Theorem. Let $[a_0; a_1, a_2, \dots]$ be an infinite continued fraction with $a_k > 0$ for $k \geq 1$. Then $[a_0; a_1, a_2, \dots]$ is irrational.

Proof. Write $x = [a_0; a_1, a_2, \dots]$ for short. I want to show that x is irrational. Suppose on the contrary that $x = \frac{p}{q}$, where p and q are integers. I will show this leads to a contradiction.

Since the odd convergents are bigger than x and the even convergents are smaller than x ,

$$c_{2k+1} > x > c_{2k}.$$

Then

$$\begin{aligned} c_{2k+1} - c_{2k} &> x - c_{2k} > 0, \\ \frac{(-1)^{2k}}{q_{2k}q_{2k+1}} &> x - c_{2k} > 0, \\ \frac{1}{q_{2k}q_{2k+1}} &> x - c_{2k} > 0, \\ \frac{1}{q_{2k}q_{2k+1}} &> x - \frac{p_{2k}}{q_{2k}} > 0, \\ \frac{1}{q_{2k+1}} &> xq_{2k} - p_{2k} > 0, \\ \frac{1}{q_{2k+1}} &> \frac{pq_{2k}}{q} - p_{2k} > 0, \\ \frac{q}{q_{2k+1}} &> pq_{2k} - p_{2k}q > 0. \end{aligned}$$

Notice that this inequality is true for all k , and that $p_{2k}q - p_{2k-1}q$ is an *integer*. But q is fixed, and $q_{2k+1} \geq 2k + 1$, so if I make k sufficiently large eventually q_{2k+1} will become bigger than q . Then $\frac{q}{q_{2k+1}}$ will be a fraction less than 1, and I have an *integer* $p_{2k}q - p_{2k-1}q$ caught between 0 and a fraction less than 1. Since this is impossible, x can't be rational. \square

Now I know that every infinite continued fraction made of positive integers represents an irrational number. The converse is also true, and it gives an algorithm for computing the continued fraction expansion.

Proposition Let $x \in \mathbb{R}$ be irrational. Let $x_0 = x$, and

$$a_k = [x_k], \quad x_{k+1} = \frac{1}{x_k - a_k} \quad \text{for } k \geq 0.$$

Then for all $k \geq 0$,

$$x = [a_0; a_1, a_2, \dots, x_k] = a_0 + \frac{1}{a_1 + \frac{1}{\ddots \frac{1}{a_{k-1} + \frac{1}{x_k}}}}$$

Thus, x_k represents the “infinite tail” of the continued fraction.

Proof. For $k = 0$, the claim is that $x = x_0$, which is true by definition.

Assume that the result holds for n . Then

$$\begin{aligned} x_{k+1} &= \frac{1}{x_k - a_k} \\ x_k - a_k &= \frac{1}{x_{k+1}} \\ x_k &= a_k + \frac{1}{x_{k+1}} \end{aligned}$$

Then

$$x = [a_0; a_1, a_2, \dots, x_k] = \left[a_0; a_1, a_2, \dots, a_k + \frac{1}{x_{k+1}} \right] = [a_0; a_1, a_2, \dots, a_k, x_{k+1}].$$

This proves the result for $k + 1$, so the result is true for $k \geq 0$ by induction. \square

Theorem. Let $x \in \mathbb{R}$ be irrational. Let $x_0 = x$, and

$$a_k = [x_k], \quad x_{k+1} = \frac{1}{x_k - a_k} \quad \text{for } k \geq 0.$$

Then

$$x = [a_0; a_1, a_2, \dots].$$

Proof.

Step 1. x_k is irrational for $k \geq 0$.

Since x is irrational and $x_0 = x$, the result is true for $k = 0$.

Assume that $k > 0$ and that the result is true for $k - 1$. I want to show that x_k is irrational.

Suppose on the contrary that $x_k = \frac{s}{t}$, where $s, t \in \mathbb{Z}$. Then

$$\frac{s}{t} = \frac{1}{x_{k-1} - a_{k-1}} \quad \text{so} \quad x_{k-1} = a_{k-1} + \frac{t}{s}.$$

Now all the a_k 's are clearly integers (since $a_k = [x_k]$ means they're outputs of the greatest integer function), so $a_{k-1} + \frac{t}{s}$ is the sum of an integer and a rational number. Therefore, it's rational, so x_{k-1} is rational, contrary to the induction hypothesis.

It follows that x_k is irrational. By induction, x_k is irrational for all $k \geq 0$.

Step 2. The a_k 's are positive integers for $k \geq 1$.

I already observed that the a_k 's are integers.

Let $k \geq 0$. Since $a_k = [x_k]$, the definition of the greatest integer function gives

$$a_k \leq x_k < a_k + 1.$$

But x_k is irrational, so $a_k \neq x_k$. Hence,

$$\begin{aligned} a_k &< x_k < a_k + 1 \\ 0 &< x_k - a_k < 1 \\ x_{k+1} &= \frac{1}{x_k - a_k} > 1 \\ a_{k+1} &= [x_{k+1}] \geq 1 \end{aligned}$$

Since $k \geq 0$, this proves that the a_k 's are positive integers for $k \geq 1$.

Step 3.

$$\lim_{k \rightarrow \infty} c_k = \lim_{k \rightarrow \infty} [a_0; a_1, \dots, a_k] = x.$$

First, I'll get a formula for x in terms of the p 's, q 's, and a 's.

Then I'll find $\left| x - \frac{p_k}{q_k} \right|$ and show that it's less than something which goes to 0.

Recall the recursion formulas for convergents:

$$p_k = a_k p_{k-1} + p_{k-2} \quad \text{and} \quad q_k = a_k q_{k-1} + q_{k-2}.$$

The right sides only involve terms up to a_k (and p 's and q 's of smaller indices). Therefore, the following fractions have the same p 's and q 's through index k :

$$[a_0; a_1, a_2, \dots, a_k, x_{k+1}] \quad \text{and} \quad [a_0; a_1, a_2, \dots, a_k, a_{k+1}, \dots].$$

Using the preceding proposition and the recursion formula for convergents, I get

$$x = x_0 = [a_0; a_1, a_2, \dots, a_k, x_{k+1}] = \frac{x_{k+1} p_k + p_{k-1}}{x_{k+1} q_k + q_{k-1}}.$$

Therefore,

$$\begin{aligned} x - \frac{p_k}{q_k} &= \frac{x_{k+1} p_k + p_{k-1}}{x_{k+1} q_k + q_{k-1}} - \frac{p_k}{q_k} = \frac{x_{k+1} p_k q_k + p_{k-1} q_k - x_{k+1} p_k q_k - p_k q_{k-1}}{(x_{k+1} q_k + q_{k-1}) q_k} = \\ &= \frac{p_{k-1} q_k - p_k q_{k-1}}{(x_{k+1} q_k + q_{k-1}) q_k} = \frac{(-1)^k}{(x_{k+1} q_k + q_{k-1}) q_k}. \end{aligned}$$

Take absolute values:

$$\left| x - \frac{p_k}{q_k} \right| = \frac{1}{(x_{k+1}q_k + q_{k-1})q_k}.$$

Now

$$x_{k+1} > [x_{k+1}] = a_{k+1}, \quad \text{so} \quad x_{k+1}q_k + q_{k-1} > a_{k+1}q_k + q_{k-1} = q_{k+1}.$$

Therefore,

$$\begin{aligned} \frac{1}{x_{k+1}q_k + q_{k-1}} &< \frac{1}{q_{k+1}}, \\ \frac{1}{(x_{k+1}q_k + q_{k-1})q_k} &< \frac{1}{q_{k+1}q_k}, \\ \left| x - \frac{p_k}{q_k} \right| &< \frac{1}{q_{k+1}q_k}. \end{aligned}$$

By an earlier lemma, $q_k \geq k$ and $q_{k+1} \geq k+1$, so

$$\left| x - \frac{p_k}{q_k} \right| < \frac{1}{q_{k+1}q_k} \leq \frac{1}{k(k+1)}.$$

Now $\lim_{k \rightarrow \infty} \frac{1}{k(k+1)} = 0$, so by the Squeezing Theorem

$$\lim_{k \rightarrow \infty} \left| x - \frac{p_k}{q_k} \right| = 0.$$

This implies that

$$\lim_{k \rightarrow \infty} \frac{p_k}{q_k} = x. \quad \square$$

Example. Compute the first 5 terms of the continued fraction expansion of π .

Here are the first two steps:

$$\begin{aligned} x_0 &= \pi, \quad a_0 = [x_0] = [\pi] = 3 \\ x_1 &= \frac{1}{x_0 - a_0} \approx 7.06251, \quad a_1 = [x_1] = 7 \end{aligned}$$

Continuing in this way, I obtain:

x_k	a_k	p_k	q_k	c_k
π	3	3	1	3
7.06251...	7	22	7	$\frac{22}{7}$
15.99659...	15	333	106	$\frac{333}{106}$
1.00341...	1	355	113	$\frac{355}{113}$
292.63459...	292	103993	33102	$\frac{103993}{33102}$

\square

Example. Compute the first 5 terms of the continued fraction expansion of $\sqrt{5}$.

$$x_0 = \sqrt{5} \quad a_0 = [\sqrt{5}] \approx [2.23607] = 2$$

$$x_1 = \frac{1}{\sqrt{5} - 2}, \quad a_1 = \left[\frac{1}{\sqrt{5} - 2} \right] \approx [4.23607] = 4$$

Here are the first 5 terms:

x	a
$\sqrt{5}$	2
4.23606...	4
4.23606...	4
4.23606...	4
4.23606...	4

In fact, the continued fraction expansion for $\sqrt{5}$ is $[2; 4, 4, 4, \dots]$. \square

Theorem. The continued fraction expansion of an irrational number is unique.

Proof. Suppose there are two continued fractions for the irrational number x , where $a_k, b_k \in \mathbb{Z}$ and $a_k, b_k \geq 1$ for $k \geq 1$:

$$[a_0; a_1, a_2, \dots] = x = [b_0; b_1, b_2, \dots].$$

I want to show that $a_k = b_k$ for all k .

Recall that the even convergents are smaller than x and the odd convergents are greater than x . Therefore,

$$a_0 < x < a_0 + \frac{1}{a_1}.$$

Now

$$a_1 \geq 1$$

$$\frac{1}{a_1} \leq 1$$

$$a_0 + \frac{1}{a_1} \leq a_0 + 1$$

$$x < a_0 + 1$$

Thus, a_0 is an integer less than x , and the next larger integer $a_0 + 1$ is greater than x . This means that $a_0 = [x]$.

The same reasoning applies to the b 's. Therefore, $b_0 = [x]$, so $a_0 = b_0$.

Hence,

$$[a_0; a_1, a_2, \dots] = [b_0; b_1, b_2, \dots]$$

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}} = b_0 + \frac{1}{b_1 + \frac{1}{b_2 + \dots}}$$

$$\frac{1}{a_1 + \frac{1}{a_2 + \dots}} = \frac{1}{b_1 + \frac{1}{b_2 + \dots}}$$

$$[a_1; a_2, a_3, \dots] = [b_1; b_2, b_3, \dots]$$

I can continue in the same way to show that $a_k = b_k$ for all k . \square

Here's a summary of some of the important results on infinite continued fractions:

1. An irrational number has a unique infinite continued fraction expansion.
2. The algorithm for computing the continued fraction expansion of an irrational number x is:

$$x_0 = x, \quad \text{and} \quad a_k = [x_k], \quad x_{k+1} = \frac{1}{x_k - a_k} \quad \text{for } k \geq 1.$$

Then

$$x = [a_0; a_1, a_2, \dots].$$

3. If $[a_0; a_1, a_2, \dots]$ is the continued fraction expansion of an irrational number, then a_0 is an integer, and a_k is a positive integer for $k \geq 1$.
4. If $[a_0; a_1, a_2, \dots]$ is a continued fraction expansion and $a_k > 0$ for $k \geq 1$, then $[a_0; a_1, a_2, \dots]$ is irrational.
5. If $x = [a_0; a_1, a_2, \dots]$ is the continued fraction expansion of an irrational number and $\{p_k\}$ and $\{q_k\}$ are defined by the recursion formulas for convergents, then

$$\left| x - \frac{p_k}{q_k} \right| < \frac{1}{q_{k+1}q_k}.$$

Example. Compute the first 6 convergents in the continued fraction expansion for $e + \pi$.

Use them to illustrate the last result for $k = 4$.

x_k	a_k	p_k	q_k	c_k
5.85987...	5	5	1	5
1.16296...	1	6	1	6
6.13646...	6	41	7	$\frac{41}{7}$
7.32821...	7	293	50	$\frac{293}{50}$
3.0468...	3	920	157	$\frac{920}{157}$
21.3697...	21	19613	3347	$\frac{19613}{3347}$

Now

$$e + \pi - \frac{920}{157} \approx 0.000001871 \quad \text{while} \quad \frac{1}{157 \cdot 3347} \approx 0.000001903.$$

Thus, in this case,

$$\left| e + \pi - \frac{920}{157} \right| < \frac{1}{157 \cdot 3347}. \quad \square$$

[1] Walter Rudin, *Principles of Mathematical Analysis* (3rd edition). New York: McGraw-Hill Book Company, 1976.