Purely Periodic Continued Fractions

We've seen that quadratic irrationals correspond to periodic continued fractions. A periodic continued fraction may repeat eventually (like $[1, 2, 3, 4, 3, 4, \ldots]$) or repeat from the start (like $[3, 4, 3, 4, \ldots]$). In this section, I'll consider the second case.

Definition. A continued fraction of the form $[\overline{a_0, a_1, \ldots a_n}]$ is **purely periodic**.

I'll derive a criterion for a quadratic irrational to have a purely periodic continued fraction. It is a result of Galois from 1829 ([1]). Recall that if $x = \dfrac{a + b\sqrt{d}}{c}$ is a quadratic irrational, its conjugate is

$$\overline{x} = \frac{a - b\sqrt{d}}{c}.$$

Theorem. Let $x = [a_0, a_1, \ldots]$ be a quadratic irrational. The continued fraction for x is purely periodic if and only if $x > 1$ and $-1 < \overline{x} < 0$.

Proof. $(\leftarrow)$ Suppose $x > 1$ and $-1 < \overline{x} < 0$. Using the general continued fraction algorithm and properties of conjugates, I have

$$x_{n+1} = \frac{1}{x_n - a_n}$$
$$\frac{1}{x_{n+1}} = x_n - a_n$$
$$\frac{1}{\overline{x_{n+1}}} = \overline{x_n} - a_n$$

Note that $a_n \geq 1$ for $n \geq 1$. But $x_0 = x > 1$, so $a_0 \geq 1$. Thus, $a_n \geq 1$ for $n \geq 0$.

Claim: For all $n \geq 0$,

$$-1 < \overline{x_n} < 0.$$

I'll prove the claim using induction. First, $\overline{x} = \overline{x_0}$, so by assumption $-1 < \overline{x_0} < 0$.
Assume that $-1 < \overline{x_n} < 0$. Then $a_n \geq 1$ gives $-a_n \leq -1$, so adding $\overline{x_n} < 0$ gives

$$\overline{x_n} - a_n < -1$$
$$\frac{1}{\overline{x_{n+1}}} < -1$$
$$\overline{x_{n+1}} > -1$$

Since the middle inequality shows $\overline{x_{n+1}} < 0$, I have

$$-1 < \overline{x_{n+1}} < 0.$$

This proves the claim by induction.

Now

$$\overline{x_n} - a_n = \frac{1}{\overline{x_{n+1}}}$$
$$\overline{x_n} = \frac{1}{\overline{x_{n+1}}} + a_n$$

Using the claim, I have

$$-1 < \frac{1}{\overline{x_{n+1}}} + a_n < 0$$
$$1 > \left(-\frac{1}{\overline{x_{n+1}}}\right) - a_n > 0$$

This inequality says that $-\frac{1}{\overline{x_{n+1}}} > a_n$, and also that a_n is an integer which differs from $-\frac{1}{\overline{x_{n+1}}}$ by less than 1. It follows that

$$a_n = \left[-\frac{1}{\overline{x_{n+1}}}\right].$$

Since x is a quadratic irrational, its continued fraction is periodic. Thus, there are indices i and j such that

$$x_i = x_j \quad \text{where} \quad 0 < i < j.$$

Hence,

$$\begin{aligned} \overline{x_i} &= \overline{x_j} \\ \left[-\frac{1}{\overline{x_i}}\right] &= \left[-\frac{1}{\overline{x_j}}\right] \\ a_{i-1} &= a_{j-1} \\ a_{i-1} + \frac{1}{x_i} &= a_{j-1} + \frac{1}{x_j} \\ x_{i-1} &= x_{j-1} \end{aligned}$$

Thus, $x_i = x_j$ implies $x_{i-1} = x_{j-1}$. Continuing to reduce indices in this way, I eventually obtain

$$x_0 = x_{j-i}.$$

Therefore,

$$x = x_0 = [\overline{a_0, a_1, \ldots, a_{j-i-1}}].$$

Hence, x is purely periodic.

($\rightarrow$) Suppose x is a quadratic irrational that is purely periodic, so $x = [\overline{a_0, \ldots, a_{n-1}}]$, where $a_0, \ldots, a_{n-1} \in \mathbb{Z}^+$. Note that $x > a_0 \geq 1$. I have

$$x_n = x = [\overline{a_0, \ldots, a_{n-1}}].$$

Hence,

$$x = \frac{x_n p_{n-1} + p_{n-2}}{x_n q_{n-1} + q_{n-2}} = \frac{x p_{n-1} + p_{n-2}}{x q_{n-1} + q_{n-2}}.$$

So

$$\begin{aligned} q_{n-1}x^2 + q_{n-2}x &= p_{n-1}x + p_{n-2} \\ q_{n-1}x^2 + (q_{n-2} - p_{n-1})x - p_{n-2} &= 0 \end{aligned}$$

The quadratic function $f(t) = q_{n-1}t^2 + (q_{n-2} - p_{n-1})t - p_{n-2}$ has x and $\overline{x}$ as its roots. I already know $x > 1$; I need to show $-1 < \overline{x} < 0$. It's enough to show that f has a root between -1 and 0: Since that root can't be x, it must be $\overline{x}$.

First, $f(0) = -p_{n-2} < 0$. Next,

$$\begin{aligned} f(-1) &= q_{n-1} + (p_{n-1} - q_{n-2}) - p_{n-2} \\ &= (a_{n-1}q_{n-2} + q_{n-3}) + (a_{n-1}p_{n-2} + p_{n-3}) - p_{n-2} - q_{n-2} \\ &= (a_{n-1} - 1)p_{n-2} + (a_{n-1} - 1)q_{n-2} + p_{n-3} + q_{n-3} \\ &= (a_{n-1} - 1)(p_{n-2} + q_{n-2}) + (p_{n-3} + q_{n-3}) \\ &> p_{n-3} + q_{n-3} \\ &> 0 \end{aligned}$$

Then $f(-1) > 0$ and $f(0) < 0$ implies that there's a root between -1 and 0 by the Intermediate Value Theorem. As noted above, that root must be $\overline{x}$.

Thus, $x > 1$ and $-1 < \overline{x} < 0$. □

For example, $\dfrac{1+\sqrt{3}}{2}$ satisfies $\dfrac{1+\sqrt{3}}{2} > 1$ and $-1 < \dfrac{1-\sqrt{3}}{2} < 0$. Its continued fraction is

$$\frac{1+\sqrt{3}}{2} = [\overline{1,2}].$$

On the other hand, $\sqrt{3} > 1$, but $-\sqrt{3}$ does not lie between -1 and 0. Its continued fraction is

$$\sqrt{3} = [1, \overline{1,2}].$$

To motivate the next result, consider the following example.

Example. (a) Compute the numerators and denominators of the convergents for $[2, 1, 3, 1, 2, 4]$.

(b) Compute the numerators and denominators of the convergents for $[4, 2, 1, 3, 1, 2]$.

(a)

a	p	q
2	2	1
1	3	1
3	11	4
1	14	5
2	39	14
4	170	61

□

(b)

a	p	q
4	4	1
2	9	2
1	13	3
3	48	11
1	61	14
2	170	39

□

Look at the numbers in the last two rows of the tables in the last example. They suggest the following result.

Theorem. Consider the continued fractions

$$x = [a_0, a_1, \ldots a_{n-1}, a_n] \quad \text{and} \quad y = [a_n, a_{n-1}, \ldots a_1, a_0].$$

Let p_k and q_k denote the numerator and denominator of the k^{th} convergent for x.
Let p'_k and q'_k denote the numerator and denominator of the k^{th} convergent for y.
Then:

(a)

$$\frac{p_n}{p_{n-1}} = \frac{p'_n}{q'_n} \quad \text{and} \quad \frac{q_n}{q_{n-1}} = \frac{p'_{n-1}}{q'_{n-1}} \quad \text{for} \quad n \geq 1.$$

(b)

$$p_n = p'_n, \quad p_{n-1} = q'_n, \quad q_n = p'_{n-1}, \quad q_{n-1} = q'_{n-1}.$$

Remark. By reversing the roles of x and y, it also follows that

$$\frac{p'_n}{p'_{n-1}} = \frac{p_n}{q_n} \quad \text{and} \quad \frac{q'_n}{q'_{n-1}} = \frac{p_{n-1}}{q_{n-1}}.$$

Proof. (a) We'll induct on n. For $n = 1$, consider the convergents tables for $[a_0, a_1]$ and $[a_1, a_0]$:

a	p	q
a_0	a_0	1
a_1	$a_0a_1 + 1$	a_1

a	p	q
a_1	a_1	1
a_0	$a_0a_1 + 1$	a_0

Then

$$\frac{p_1}{p_0} = \frac{a_0a_1 + 1}{a_0} = \frac{p'_1}{q'_1},$$

$$\frac{q_1}{q_0} = a_1 = \frac{p'_0}{q'_0}.$$

The result holds for $n = 1$.

Assume that the result holds for n (that is, that it holds for a fraction with $n+1$ terms and its reverse). I need to prove the result for $n+1$ — that is, for the fractions $[a_0, a_1, \ldots a_n, a_{n+1}]$ and $[a_{n+1}, a_n, \ldots a_1, a_0]$, I have

$$\frac{p_{n+1}}{p_n} = \frac{p'_{n+1}}{q'_{n+1}} \quad \text{and} \quad \frac{q_{n+1}}{q_n} = \frac{p'_n}{q'_n}.$$

Note that the primed p's and q's are for $[a_{n+1}, a_n, \ldots a_1, a_0]$, not for $[a_n, \ldots a_1, a_0]$.

I have

$$\frac{p_{n+1}}{p_n} = \frac{a_{n+1}p_n + p_{n-1}}{p_n} = a_{n+1} + \frac{p_{n-1}}{p_n} = a_{n+1} + \frac{1}{\left(\dfrac{p_n}{p_{n-1}}\right)}.$$

I'll apply the induction hypothesis to $[a_0, a_1, \ldots a_n]$ and $[a_n, a_{n-1}, \ldots a_0]$. Note that p_{n-1} and p_n are the same for $[a_0, a_1, \ldots a_n]$ and $[a_0, a_1, \ldots a_n, a_{n+1}]$. However, the p's and q's for $[a_{n+1}, a_n, \ldots a_1, a_0]$ and for $[a_n, \ldots a_1, a_0]$ are different, so I'll use double-primed p's and q's for $[a_n, \ldots a_1, a_0]$. With this understanding, the induction hypothesis gives

$$\frac{p_n}{p_{n-1}} = \frac{p''_n}{q''_n} = [a_n, \ldots a_1, a_0].$$

Then using the last two equations, I have

$$\begin{aligned}\frac{p_{n+1}}{p_n} &= a_{n+1} + \frac{1}{\left(\dfrac{p_n}{p_{n-1}}\right)} \\ &= a_{n+1} + \frac{1}{[a_n, \ldots a_1, a_0]} \\ &= [a_{n+1}, a_n, \ldots a_1, a_0] \\ &= \frac{p'_{n+1}}{q'_{n+1}}\end{aligned}$$

Similarly,

$$\frac{q_{n+1}}{q_n} = \frac{a_{n+1}q_n + q_{n-1}}{q_n} = a_{n+1} + \frac{q_{n-1}}{q_n} = a_{n+1} + \frac{1}{\left(\dfrac{q_n}{q_{n-1}}\right)}.$$

By induction,

$$\frac{q_n}{q_{n-1}} = [a_n, a_{n-1}, \ldots a_1].$$

So

$$\frac{q_{n+1}}{q_n} = a_{n+1} + \frac{1}{[a_n, a_{n-1}, \ldots a_1]} = [a_{n+1}, a_n, \ldots a_1] = \frac{p'_n}{q'_n}.$$

This proves the result for $n+1$, so the result holds for all $n \geq 1$ by induction.

(b) Recall that

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}.$$

It follows that $\frac{p_n}{p_{n-1}}$ and $\frac{q_n}{q_{n-1}}$ are in lowest terms. But $\frac{p'_n}{q'_n}$ and $\frac{p'_{n-1}}{q'_{n-1}}$ are convergents of a continued fraction, so they're in lowest terms as well.

$\frac{p_n}{p_{n-1}} = \frac{p'_n}{q'_n}$ is an equality between fractions in lowest terms, so $p_n = p'_n$ and $p_{n-1} = q'_n$. Likewise, $\frac{q_n}{q_{n-1}} = \frac{p'_{n-1}}{q'_{n-1}}$ is an equality between fractions in lowest terms, so $q_n = p'_{n-1}$ and $q_{n-1} = q'_{n-1}$. □

This result relates a finite continued fraction $[a_0, a_1, \ldots a_n]$ and its "reverse" $[a_n, a_{n-1}, \ldots a_0]$. The next result (also due to Galois) considers the relationship between the purely periodic continued fractions $[\overline{a_0, a_1, \ldots a_n}]$ and $[\overline{a_n, a_{n-1}, \ldots a_0}]$.

Theorem. Let $x = [\overline{a_0, a_1, \ldots a_{n-1}, a_n}]$ be a purely periodic quadratic irrational. Then $-\frac{1}{\overline{x}}$ is purely periodic, and

$$-\frac{1}{\overline{x}} = [\overline{a_n, a_{n-1}, \ldots a_1, a_0}].$$

Proof. The idea is to show that $[\overline{a_0, a_1, \ldots a_{n-1}, a_n}]$ and $-\frac{1}{[\overline{a_n, a_{n-1}, \ldots a_1, a_0}]}$ are roots of the same quadratic equation. This implies that they are conjugates.

Let $\frac{p_k}{q_k}$ be the k^{th} convergent of x. Then

$$x = [\overline{a_0, a_1, \ldots a_{n-1}, a_n}] = [a_0, a_1, \ldots a_{n-1}, a_n, x].$$

The convergents algorithm gives

$$x = \frac{p_n x + p_{n-1}}{q_n x + q_{n-1}}$$

$$q_n x^2 + q_{n-1} x = p_n x + p_{n-1}$$

$$q_n x^2 + (q_{n-1} - p_n)x - p_{n-1} = 0$$

Let

$$y = [\overline{a_n, a_{n-1}, \ldots a_1, a_0}] = [a_n, a_{n-1}, \ldots a_1, a_0, y].$$

Let $\frac{p'_k}{q'_k}$ denote the k^{th} convergent of y. The convergents algorithm gives

$$y = \frac{p'_n y + p'_{n-1}}{q'_n y + q'_{n-1}}.$$

By the preceding theorem, $p'_n = p_n$, $p'_{n-1} = q_n$, $q'_n = p_{n-1}$, and $q'_{n-1} = q_{n-1}$. So

$$\begin{aligned} y &= \frac{p_n y + q_n}{p_{n-1} y + q_{n-1}} \\ p_{n-1} y^2 + q_{n-1} y &= p_n y + q_n \\ p_{n-1} y^2 + (q_{n-1} - p_n) y - q_n &= 0 \\ p_{n-1} + (q_{n-1} - p_n) \left(\frac{1}{y}\right) - q_n \left(\frac{1}{y}\right)^2 &= 0 \\ q_n \left(-\frac{1}{y}\right)^2 + (q_{n-1} - p_n) \left(-\frac{1}{y}\right) - p_{n-1} &= 0 \end{aligned}$$

Thus, x and $-\dfrac{1}{y}$ are roots of the quadratic $q_n t^2 + (q_{n-1} - p_n)t - p_{n-1} = 0$, so they must be conjugates:

$$\begin{aligned} \overline{x} &= -\frac{1}{y} \\ y &= -\frac{1}{\overline{x}} \\ -\frac{1}{\overline{x}} &= [\overline{a_n, a_{n-1}, \ldots a_1, a_0}] \end{aligned}$$

□

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[1] É. Galois, Démonstration d'un théorème sur les fractions continues périodiques, Annalles de mathématiques 19 (1828), 294–301.