Math 211
10-2-2020

Review Problems for Test 2

These problems are provided to help you study. The presence of a problem on this sheet does not imply that a similar problem will appear on the test. And the absence of a problem from this sheet does not imply that the test will not have a similar problem.

1. Find the area of the region bounded by the graphs of $y = x^2 - 3x$ and $y = 15 - x$.

2. Find the area of the region between $y = x^2 - x$ and $y = x + 8$ from $x = 0$ to $x = 5$.

3. Find the area of the region bounded by $y = x^2 - 2x - 8$ and $y = -x^2 + 4x + 12$.

4. Find the area of the region bounded by $x = \cos y$ and $x = \sin y$, between the first two intersections of the curves for which $y > 0$.

5. The region bounded by $y = 4x - x^2$ and the x-axis is revolved about the x-axis. Find the volume of the solid that is generated.

6. Consider the region in the x-y plane bounded by $y = e^x$, the line $y = 1$, and the line $x = 1$. Find the volume generated by revolving the region:

(a) About the line $y = 1$.

(b) About the line $x = 2$.

(c) About the line $y = e$.

7. The base of a solid is the region in the x-y plane bounded by the curves $y = x^2$ and $y = x + 2$. The cross-sections of the solid perpendicular to the x-y plane and the x-axis are isosceles right triangles with one leg in the x-y plane. Find the volume of the solid.

8. The base of a solid is the region in the x-y-plane bounded above by the line $y = 1$ and below by the parabola $y = x^2$. The cross-sections in planes perpendicular to the y-axis are squares having one edge in the x-y-plane. Find the volume of the solid.

9. The region which lies above the x-axis and below the graph of $y = \dfrac{1}{x^2 + 1}$, $-\infty < x < \infty$, is revolved about the x-axis. Find the volume of the solid which is generated.

Hint:

$$\int \frac{1}{(x^2+1)^2}\,dx = \frac{1}{2}\frac{x}{x^2+1} + \frac{1}{2}\tan^{-1} x + C.$$

10. A force of 8 pounds is required to extend a spring 2 feet beyond its unstretched length.

(a) Find the spring constant k.

(b) Find the work done in stretching the spring from 2 feet beyond its unstretched length to 3 feet beyond its unstretched length.

11. The base of a rectangular tank is 2 feet long and 3 feet wide; the tank is 6 feet high. Find the work done in pumping all the water out of the top of the tank.

12. Write a formula for the n-th term of the sequence, assuming that the terms continue in the "obvious" way.

(a) $7, 11, 15, 19, 23, 27, \ldots.$

(b) $\frac{2}{8}, \frac{4}{13}, \frac{6}{18}, \frac{8}{23}, \ldots.$

13. A sequence is defined recursively by

$$a_{n+1} = 3a_n + 5 \quad \text{for} \quad n \geq 0 \quad \text{and} \quad a_0 = 1.$$

Write down the first 5 terms of the sequence.

14. Determine whether the sequence $a_n = \dfrac{e^n}{n+1}$ for $n \geq 1$ eventually increases, decreases, or neither increases nor decreases.

15. Determine whether the sequence $a_n = \cos(\pi n)$ for $n \geq 0$ eventually increases, decreases, or neither increases nor decreases.

16. Is the following sequence bounded? Why or why not?

$$1,\ 1,\ 1,\ 2,\ 1,\ 3 \ldots 1,\ n,\ \ldots.$$

17. Determine whether the sequence converges or diverges; if it converges, find the limit.

(a) $\{1.0001^n\}$.

(b) $\left\{\dfrac{e^n + 3^n}{2^n + \pi^n}\right\}$.

(c) $\left\{\dfrac{2n^3 - 5n + 7}{7n^2 - 13n^3}\right\}$.

(d) $\left\{\left(\tan^{-1} n\right)^2\right\}$.

(e) $\left\{\dfrac{\sin n}{n^2}\right\}$.

(f) $\left\{\left(\dfrac{4n+1}{9n+17} + e^{-n^2}\right)^n\right\}$.

18. A sequence is defined recursively by

$$a_1 = 5, \quad a_{n+1} = \sqrt{6a_n + 27} \quad \text{for} \quad n \geq 1.$$

Find $\lim_{n\to\infty} a_n$.

19. If the series converges, find the exact value of its sum; if it diverges, explain why.

(a) $\displaystyle\sum_{n=1}^{\infty} \left(-\frac{1}{5}\right)^n$.

(b) $\displaystyle\sum_{n=1}^{\infty} (-1.021)^n$.

(c)

$$\frac{3}{5^3} + \frac{3}{5^4} + \frac{3}{5^5} + \cdots + \frac{3}{5^n} + \cdots.$$

(d) $\displaystyle\sum_{n=2}^{\infty}\left(\frac{6^n}{7^n}+2\cdot\frac{(-1)^n}{4^n}\right)$.

(e) $\displaystyle\sum_{n=3}^{\infty}\left(\frac{5^n}{4^n}+\frac{4^n}{5^n}\right)$.

(f) $\displaystyle\sum_{n=3}^{\infty}\ln\frac{n}{n+1}$.

20. (a) Find the partial fractions decomposition of $\dfrac{2}{(2k+1)(2k+3)}$.

(b) Use (a) to find the sum of the series

$$\sum_{k=1}^{\infty}\frac{2}{(2k+1)(2k+3)}.$$

21. Find series $\displaystyle\sum_{n=1}^{\infty}a_n$ and $\displaystyle\sum_{n=1}^{\infty}b_n$ such that both series diverge, and:

(a) $\displaystyle\sum_{n=1}^{\infty}(a_n+b_n)$ diverges.

(b) $\displaystyle\sum_{n=1}^{\infty}(a_n+b_n)$ converges.

22. Calvin Butterball notes that $\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0$, and concludes that the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ converges by the Zero Limit Test. What's wrong with his reasoning?

23. If the series $\displaystyle\sum_{k=17}^{\infty}a_k$, converges, does the series $\displaystyle\sum_{k=1}^{\infty}a_k$ converge?

24. Does the series $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{2k+1}{4k+3}$ converge?

25. Determine whether the series converges or diverges: $\displaystyle\sum_{k=1}^{\infty}\frac{3^k+2^k}{6^k}$.

26. Determine whether the series converges or diverges: $\displaystyle\sum_{k=3}^{\infty}\frac{k^2-3k+2}{k^4}$.

27. Determine whether the series converges or diverges: $\displaystyle\sum_{k=1}^{\infty}\sqrt{\tan^{-1}k}$.

28. Determine whether the series converges or diverges: $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^{1.05}}$.

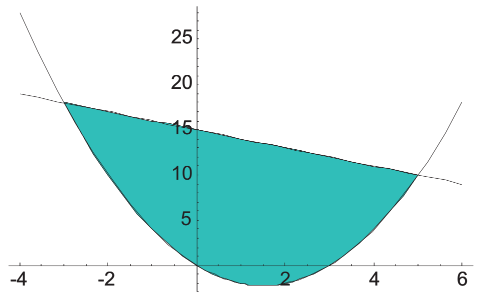
29. Determine whether the series converges or diverges: $\displaystyle\sum_{k=2}^{\infty}\frac{1}{k(\ln k)^2}$.

30. Determine whether the series converges or diverges: $\displaystyle\sum_{k=1}^{\infty} \frac{2}{3k+5}$.

31. Determine whether the series converges or diverges: $\displaystyle\sum_{k=1}^{\infty} \frac{e^x}{e^{2x}+1}$.

Solutions to the Review Problems for Test 2

1. Find the area of the region bounded by the graphs of $y = x^2 - 3x$ and $y = 15 - x$.



The curves intersect at $x = -3$ and at $x = 5$:

$$\begin{aligned} x^2 - 3x &= 15 - x \\ x^2 - 2x - 15 &= 0 \\ (x-5)(x+3) &= 0 \\ x = 5 \quad &\text{or} \quad x = -3 \end{aligned}$$

$y = 15 - x$ is the top curve and $y = x^2 - 3x$ is the bottom curve. Hence, the area is

$$\int_{-3}^{5} \left((15 - x) - (x^2 - 3x)\right)\,dx = \int_{-3}^{5} \left(15 + 2x - x^2\right)\,dx = \left[15x + x^2 - \frac{1}{3}x^3\right]_{-3}^{5} = \frac{256}{3}. \quad \square$$

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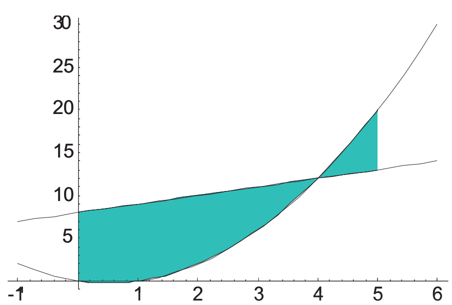
2. Find the area of the region between $y = x^2 - x$ and $y = x + 8$ from $x = 0$ to $x = 5$.



The curves intersect at $x = 4$ and $x = -2$:

$$\begin{aligned} x^2 - x &= x + 8 \\ x^2 - 2x - 8 &= 0 \\ (x-4)(x+2) &= 0 \\ x = 4 \quad &\text{or} \quad x = -2 \end{aligned}$$

Since the curves cross between 0 and 5, I will need two integrals. On the left-hand piece, the top curve is $y = x + 8$ and the bottom curve is $y = x^2 - x$. On the right-hand piece, the top curve is $y = x^2 - x$ and the bottom curve is $y = x + 8$. The area is

$$\int_0^4 \left((x+8) - (x^2 - x)\right)\, dx + \int_4^5 \left((x^2 - x) - (x+8)\right)\, dx = \int_0^4 (-x^2 + 2x + 8)\, dx + \int_4^5 (x^2 - 2x - 8)\, dx =$$

$$\left[-\frac{1}{3}x^3 + x^2 + 8x\right]_0^4 + \left[\frac{1}{3}x^3 - x^2 - 8x\right]_4^5 = 30. \quad \square$$

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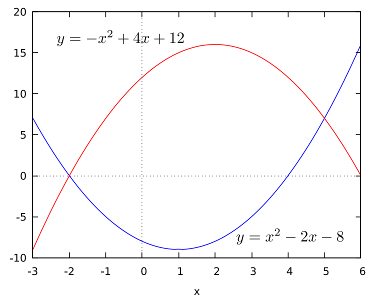
3. Find the area of the region bounded by $y = x^2 - 2x - 8$ and $y = -x^2 + 4x + 12$.



$$\begin{aligned} x^2 - 2x - 8 &= -x^2 + 4x + 12 \\ 2x^2 - 6x - 20 &= 0 \\ x^2 - 3x - 10 &= 0 \\ (x-5)(x+2) &= 0 \end{aligned}$$

The curves intersect at $x = 5$ and $x = -2$. The top curve is $y = -x^2 + 4x + 12$ and the bottom curve is $y = x^2 - 2x - 8$. The area is

$$\int_{-2}^5 \left((-x^2 + 4x + 12) - (x^2 - 2x - 8)\right)\, dx = \int_{-2}^5 (-2x^2 + 6x + 20)\, dx = \left[-\frac{2}{3}x^3 + 3x^2 + 20x\right]_{-2}^5 =$$

$$\frac{343}{3} = 114.33333\ldots. \quad \square$$

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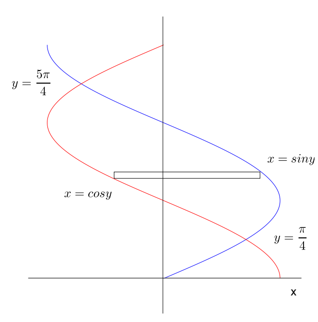
4. Find the area of the region bounded by $x = \cos y$ and $x = \sin y$, between the first two intersections of the curves for which $y > 0$.



Solve the curve equations simultaneously:

$$\begin{aligned} \sin y &= \cos y \\ \tan y &= 1 \\ y &= \frac{\pi}{4}, \ \frac{5\pi}{4} \end{aligned}$$

Break the region up into horizontal rectangles. The length of a typical rectangle is $\sin y - \cos y$. The area is

$$\int_{\pi/4}^{5\pi/4} (\sin y - \cos y)\, dy = [-\cos y - \sin y]_{\pi/4}^{5\pi/4} = 2\sqrt{2} = 2.82842\ldots. \quad \square$$

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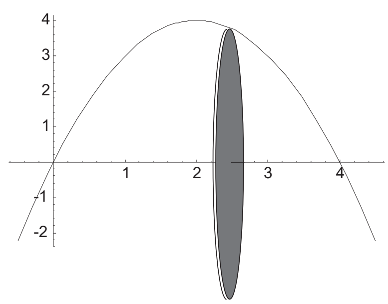
5. The region bounded by $y = 4x - x^2$ and the x-axis is revolved about the x-axis. Find the volume of the solid that is generated.



The region extends from $x = 0$ to $x = 4$. I'll use circular slices. The radius of a typical slice is $r = y = 4x - x^2$. The area of a typical slice is

$$\pi r^2 = \pi(4x - x^2)^2 = \pi(16x^2 - 8x^3 + x^4).$$

The volume generated is

$$V = \int_0^4 \pi(16x^2 - 8x^3 + x^4)\, dx = \pi \left[\frac{16}{3}x^3 - 2x^4 + \frac{1}{5}x^5\right]_0^4 = \frac{512\pi}{15} = 107.23302\ldots. \quad \square$$

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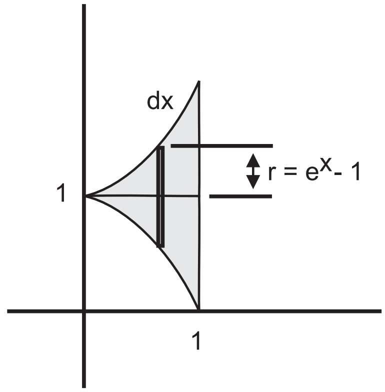
6. Consider the region in the x-y plane bounded by $y = e^x$, the line $y = 1$, and the line $x = 1$. Find the volume generated by revolving the region:

(a) About the line $y = 1$.

(b) About the line $x = 2$.

(c) About the line $y = e$.

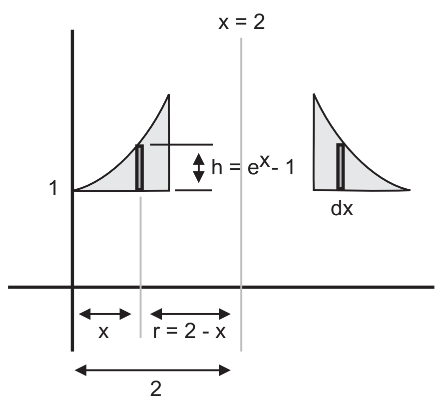
(a)



Since the solid has no "holes" or "gaps" in its interior, I can use circular slices. The radius of a slice is $r = e^x - 1$, so the volume is

$$V = \int_0^1 \pi(e^x - 1)^2\,dx = \pi \int_0^1 (e^{2x} - 2e^x + 1)\,dx = \pi\left[\frac{1}{2}e^{2x} - 2e^x + x\right]_0^1 = \frac{\pi e^2}{2} - 2\pi e + \frac{5\pi}{2} = 2.38121\ldots. \quad \square$$

(b)



I'll use cylindrical shells. The height is $h = e^x - 1$, and the radius is $r = 2 - x$. The volume is

$$V = \int_0^1 2\pi(e^x - 1)(2 - x)\,dx = 2\pi \int_0^1 (2e^x - 2 - xe^x + x)\;dx = 2\pi\left[2e^x - 2x - xe^x + e^x + \frac{1}{2}x^2\right]_0^1 =$$

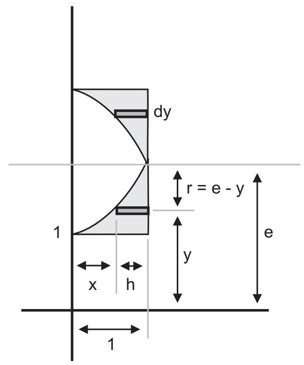
$$4\pi e - 9\pi = 5.88460\ldots.$$

Here's the work for part of the integral:

	$\frac{d}{dx}$		$\int dx$
$+$	x	$\searrow$	e^x
$-$	1	$\searrow$	e^x
$+$	0	$\rightarrow$	e^x

$$\int xe^x\,dx = xe^x - e^x + C. \quad \square$$

(c)



I'll use cylindrical shells. Since $y = e^x$ gives $x = \ln y$, the height is $h = 1 - x = 1 - \ln y$, and the radius is $r = e - y$. The vertical limits on the region are $y = 1$ and $y = e$. The volume is

$$V = \int_1^e 2\pi(1 - \ln y)(e - y)\,dy = 2\pi \int_1^e (e - e\ln y - y + y\ln y)\,dy =$$

$$2\pi\left[ey - ey\ln y + ey - \frac{1}{2}y^2 + \frac{1}{2}y^2\ln y - \frac{1}{4}y^2\right]_1^e = \frac{3\pi e^2}{2} - 4\pi e + \frac{3\pi}{2} = 5.37355\ldots.$$

Here is how I did two of the pieces of the integral:

	$\dfrac{d}{dy}$		$\int dy$
$+$	$\ln y$	$\searrow$	1
$-$	$\dfrac{1}{y}$	$\rightarrow$	y

$$\int \ln y\,dy = y\ln y - \int dy = y\ln y - y + C.$$

	$\dfrac{d}{dy}$		$\int dy$
$+$	$\ln y$	$\searrow$	y
$-$	$\dfrac{1}{y}$	$\rightarrow$	$\dfrac{1}{2}y^2$

$$\int y\ln y\,dy = \frac{1}{2}y^2\ln y - \frac{1}{2}\int y\,dy = \frac{1}{2}y^2\ln y - \frac{1}{4}y^2 + C. \quad \square$$

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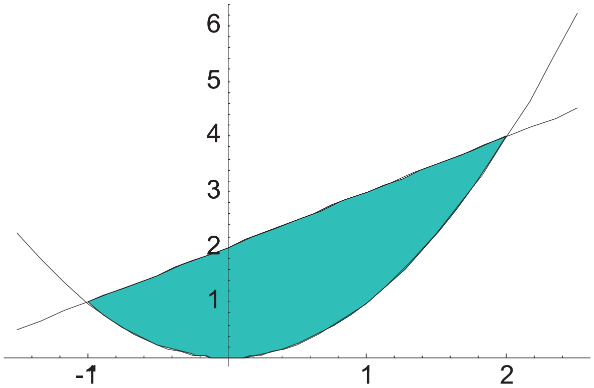
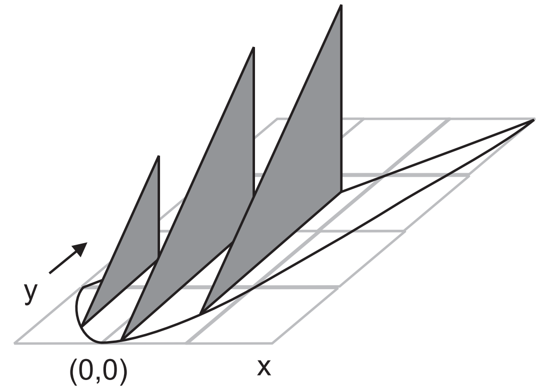
7. The base of a solid is the region in the x-y plane bounded by the curves $y = x^2$ and $y = x + 2$. The cross-sections of the solid perpendicular to the x-y plane and the x-axis are isosceles right triangles with one leg in the x-y plane. Find the volume of the solid.





The first picture shows the base of the solid. The second picture shows three typical triangular slices standing on the base.

$$\begin{aligned} x^2 &= x + 2 \\ x^2 - x - 2 &= 0 \\ (x - 2)(x + 1) &= 0 \\ x = 2 \quad &\text{or} \quad x = -1 \end{aligned}$$

Therefore, the base of the solid extends from $x = -1$ to $x = 2$.

The leg of a triangular slice has length $x+2-x^2$. Hence, the area of a triangular slice is $\dfrac{1}{2}(x+2-x^2)^2$. The volume is

$$V = \int_{-1}^{2} \frac{1}{2}(x + 2 - x^2)^2\,dx = \frac{1}{2}\int_{-1}^{2} \left(x^4 - 2x^3 - 3x^2 + 4x + 4\right)\,dx =$$

$$\frac{1}{2}\left[\frac{1}{5}x^5 - \frac{1}{2}x^4 - x^3 + 2x^2 + 4x\right]_{-1}^{2} = \frac{81}{20} = 4.05. \quad \square$$

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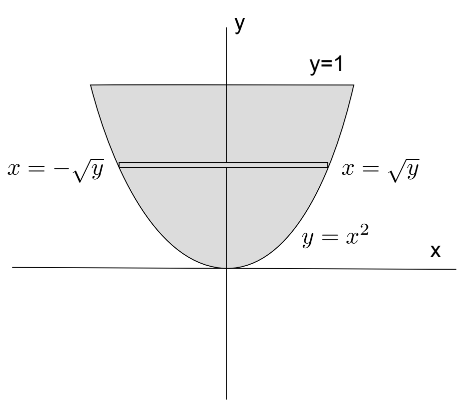
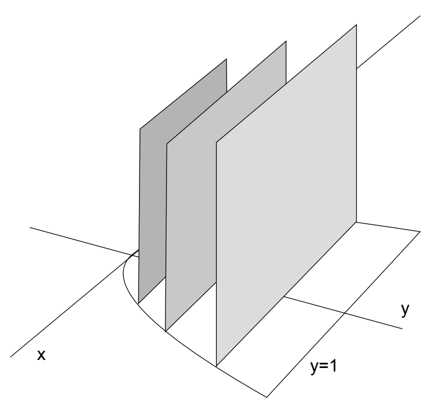
8. The base of a solid is the region in the x-y-plane bounded above by the line $y = 1$ and below by the parabola $y = x^2$. The cross-sections in planes perpendicular to the y-axis are squares having one edge in the x-y-plane. Find the volume of the solid.





The first picture shows the base of the solid. The second picture shows three typical square slices standing on the base.

The thickness of a typical slice is in the y-direction, so I'll use y as my variable. Solving $y = x^2$ for x gives $x = \pm\sqrt{y}$.

The side of a square slice extends from $x = -\sqrt{y}$ to $x = \sqrt{y}$, so its length is $\sqrt{y} - (-\sqrt{y}) = 2\sqrt{y}$. The area of a typical square slice is $(2\sqrt{y})^2 = 4y$. Hence, the volume is

$$\int_0^1 4y\,dy = \left[2y^2\right]_0^1 = 2. \quad \square$$

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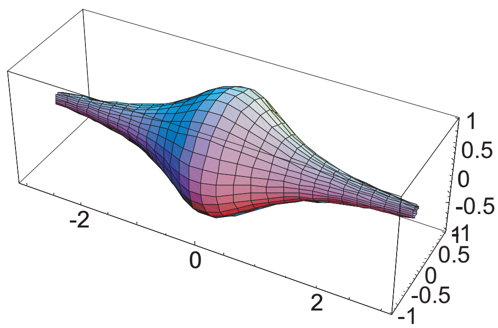
9. The region which lies above the x-axis and below the graph of $y = \dfrac{1}{x^2+1}$, $-\infty < x < \infty$, is revolved about the x-axis. Find the volume of the solid which is generated.



Chop the solid up into circular slices perpendicular to the x-axis. The thickness of a typical slice is dx. The radius of a slice is $r = \dfrac{1}{x^2+1}$. The volume is

$$V = \int_{-\infty}^{\infty} \pi \cdot \frac{1}{(x^2+1)^2}\,dx = \int_0^{\infty} \pi \cdot \frac{1}{(x^2+1)^2}\,dx + \int_{-\infty}^{0} \pi \cdot \frac{1}{(x^2+1)^2}\,dx.$$

Compute the first integral:

$$\int_0^{\infty} \pi \cdot \frac{1}{(x^2+1)^2}\,dx = \lim_{a\to+\infty} \int_0^{a} \pi \cdot \frac{1}{(x^2+1)^2}\,dx = \pi \cdot \lim_{a\to+\infty} \left[\frac{1}{2}\frac{x}{x^2+1} + \frac{1}{2}\tan^{-1} x\right]_0^a =$$

$$\frac{\pi}{2} \lim_{a\to+\infty} \left(\frac{a}{a^2+1} + \tan^{-1} a\right) = \frac{\pi^2}{4}.$$

(I used the fact that $\lim_{a\to+\infty} \tan^{-1} a = \dfrac{\pi}{2}$.)

Similarly,

$$\int_{-\infty}^{0} \pi \cdot \frac{1}{(x^2+1)^2}\,dx = \frac{\pi^2}{4}.$$

The volume is $\dfrac{\pi^2}{4} + \dfrac{\pi^2}{4} = \dfrac{\pi^2}{2}$. $\square$

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10. A force of 8 pounds is required to extend a spring 2 feet beyond its unstretched length.

(a) Find the spring constant k.

(b) Find the work done in stretching the spring from 2 feet beyond its unstretched length to 3 feet beyond its unstretched length.

(a)

$$F = -kx$$
$$8 = -2k$$
$$k = -4 \quad \square$$

(b) Since $k = -4$, I have $F = 4x$. Hence, the work done is

$$\int_2^3 4x\,dx = \left[2x^2\right]_2^3 = 10 \text{ foot-pounds.} \quad \square$$

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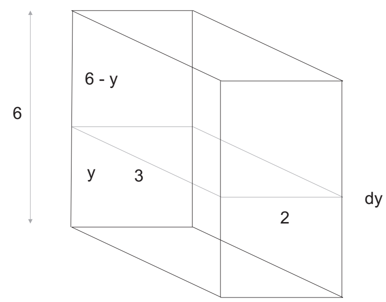
11. The base of a rectangular tank is 2 feet long and 3 feet wide; the tank is 6 feet high. Find the work done in pumping all the water out of the top of the tank.

Divide the water up into rectangular slabs parallel to the base. Let y denote the height of a slab above the base.



The volume of a typical slab is $(2)(3)\,dy = 6\,dy$, so the weight is $62.4 \cdot 6\,dy$. (The density of water is 62.4 pounds per cubic foot.)

A slab at height y must be lifted a distance of $6 - y$ to get to the top of the tank. Therefore, the work done in lifting the slab is $62.4 \cdot 6(6 - y)\,dy$. The total work is

$$\int_0^6 62.4 \cdot 6(6 - y)\,dy = 62.4 \cdot 6\left[6y - \frac{1}{2}y^2\right]_0^6 = 6739.2 \text{ foot-pounds.} \quad \square$$

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12. Write a formula for the n-th term of the sequence, assuming that the terms continue in the "obvious" way.

(a) $7, 11, 15, 19, 23, 27, \ldots$.

(b) $\dfrac{2}{8}, \dfrac{4}{13}, \dfrac{6}{18}, \dfrac{8}{23}, \ldots$.

(a)

$$a_n = 7 + 4n \quad \text{for} \quad n = 0, 1, 2, \ldots. \quad \square$$

(b)

$$a_n = \frac{2n}{3 + 5n} \quad \text{for} \quad n = 1, 2, 3, \ldots. \quad \square$$

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13. A sequence is defined recursively by

$$a_{n+1} = 3a_n + 5 \quad \text{for} \quad n \geq 0 \quad \text{and} \quad a_0 = 1.$$

Write down the first 5 terms of the sequence.

$$a_0 = 1, \quad a_1 = 8, \quad a_2 = 29, \quad a_3 = 92, \quad a_4 = 281. \quad \square$$

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14. Determine whether the sequence $a_n = \dfrac{e^n}{n+1}$ for $n \geq 1$ eventually increases, decreases, or neither increases nor decreases.

Let $f(x) = \dfrac{e^x}{x+1}$. Then

$$f'(x) = \frac{(x+1)e^x - e^x}{(x+1)^2} = \frac{xe^x}{(x+1)^2} > 0 \quad \text{for} \quad x \geq 1.$$

Hence, the sequence increases. $\square$

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15. Determine whether the sequence $a_n = \cos(\pi n)$ for $n \geq 0$ eventually increases, decreases, or neither increases nor decreases.

The terms are

$$1,\ -1,\ 1,\ -1,\ \ldots.$$

In fact, $\cos(\pi n) = (-1)^n$. Hence, the sequence neither increases nor decreases. $\square$

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16. Is the following sequence bounded? Why or why not?

$$1,\ 1,\ 1,\ 2,\ 1,\ 3 \ldots 1,\ n,\ \ldots.$$

The even-numbered terms have the form n for $n \geq 1$, and $\lim_{n\to\infty} n = \infty$. Hence, the sequence is not bounded. $\square$

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17. Determine whether the sequence converges or diverges; if it converges, find the limit.

(a) $\{1.0001^n\}$

(b) $\left\{\dfrac{e^n + 3^n}{2^n + \pi^n}\right\}$

(c) $\left\{\dfrac{2n^3 - 5n + 7}{7n^2 - 13n^3}\right\}$

(d) $\left\{\left(\tan^{-1} n\right)^2\right\}$

(e) $\left\{\dfrac{\sin n}{n^2}\right\}$.

(f) $\left\{\left(\dfrac{4n+1}{9n+17} + e^{-n^2}\right)^n\right\}$.

(a) Since $\{1.0001^n\}$ is a geometric sequence with ratio $r = 1.0001 > 1$,

$$\lim_{n\to\infty} 1.0001^n = +\infty. \quad \square$$

(b) Divide the top and bottom by π^n (since π^n is the biggest exponential in the fraction):

$$\lim_{n\to\infty} \frac{e^n + 3^n}{2^n + \pi^n} = \lim_{n\to\infty} \frac{\dfrac{e^n}{pi^n} + \dfrac{3^n}{\pi^n}}{\dfrac{2^n}{\pi^n} + 1} = \frac{0+0}{0+1} = 0.$$

I computed the limit using the fact that the following are geometric sequences:

$$\frac{e^n}{pi^n} = \left(\frac{e}{\pi}\right)^n, \quad \frac{3^n}{\pi^n} = \left(\frac{3}{\pi}\right)^n, \quad \text{and} \quad \frac{2^n}{\pi^n} = \left(\frac{2}{\pi}\right)^n.$$

Their ratios are all less than 1, so they go to 0 as $n \to \infty$. $\square$

(c)

$$\lim_{n\to\infty} \frac{2n^3 - 5n + 7}{7n^2 - 13n^3} = -\frac{2}{13}.$$

I did this by considering the highest powers on the top and bottom; they're both x^3, so I just looked at their coefficients. You could also do this by using L'Hôpital's rule, or by dividing the top and the bottom by x^3. $\square$

(d)

$$\lim_{n\to\infty} \left(\tan^{-1} n\right)^2 = \left(\lim_{n\to\infty} \tan^{-1} n\right)^2 = \left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{4}. \quad \square$$

(e) Note that $\lim\limits_{n\to\infty} \sin n$ is undefined, so I can't take the limit of the terms directly. Instead, I'll use the Squeezing Theorem. I have

$$\begin{array}{ccccc} -1 & \leq & \sin n & \leq & 1 \\ -\dfrac{1}{n^2} & \leq & \dfrac{\sin n}{n^2} & \leq & \dfrac{1}{n^2} \end{array}$$

Also,

$$\lim_{n\to\infty} -\frac{1}{n^2} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n^2} = 0.$$

By the Squeezing Theorem, $\lim\limits_{n\to\infty} \dfrac{\sin n}{n^2} = 0.$ $\square$

(f) Note that

$$\lim_{n\to\infty} \left(\frac{4n+1}{9n+17} + e^{-n^2}\right) = \frac{4}{9} + 0 = \frac{4}{9}.$$

Since $\lim_{n\to\infty} \left(\dfrac{4}{9}\right)^n = 0$, it follows that

$$\lim_{n\to\infty} \left(\frac{4n+1}{9n+17} + e^{-n^2}\right)^n = 0. \quad \square$$

---

18. A sequence is defined recursively by

$$a_1 = 5, \quad a_{n+1} = \sqrt{6a_n + 27} \quad \text{for} \quad n \geq 1.$$

Find $\lim_{n\to\infty} a_n$.

Taking the limit on both sides of the recursion equation, I get

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{6a_n + 27} = \sqrt{6 \lim_{n\to\infty} a_n + 27}.$$

I'm allowed to move the limit inside the square root by a standard rule for limits.
Now $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$ because both limits represent what the sequence $\{a_n\}$ is approaching. So let

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n.$$

Then

$$\begin{aligned} L &= \sqrt{6L + 27} \\ L^2 &= 6L + 27 \\ L^2 - 6L - 27 &= 0 \\ (L - 9)(L + 3) &= 0 \\ L = 9 \quad &\text{or} \quad L = -3 \end{aligned}$$

Since the sequence consists of positive numbers, it can't have a negative limit. This rules out -3. Therefore,

$$\lim_{n\to\infty} a_n = 9. \quad \Box$$

---

19. If the series converges, find the exact value of its sum; if it diverges, explain why.

(a) $\displaystyle\sum_{n=1}^{\infty} \left(-\frac{1}{5}\right)^n$.

(b) $\displaystyle\sum_{n=1}^{\infty} (-1.021)^n$.

(c)

$$\frac{3}{5^3} + \frac{3}{5^4} + \frac{3}{5^5} + \cdots + \frac{3}{5^n} + \cdots.$$

(d) $\displaystyle\sum_{n=2}^{\infty} \left(\frac{6^n}{7^n} + 2 \cdot \frac{(-1)^n}{4^n}\right)$.

(e) $\displaystyle\sum_{n=3}^{\infty} \left(\frac{5^n}{4^n} + \frac{4^n}{5^n}\right)$.

(f) $\displaystyle\sum_{n=3}^{\infty} \ln \frac{n}{n+1}$.

(a) The series converges, and

$$\sum_{n=1}^{\infty} \left(-\frac{1}{5}\right)^n = \frac{-\frac{1}{5}}{1 - \left(-\frac{1}{5}\right)} = -\frac{1}{6}. \quad \Box$$

(b) Since the ratio -1.021 is not in the interval $(-1, 1]$, the series diverges. In fact, it diverges by oscillation, as alternate partial sums approach $+\infty$ and $-\infty$. $\Box$

(c) The series converges, and

$$\frac{3}{5^3} + \frac{3}{5^4} + \frac{3}{5^5} + \cdots + \frac{3}{5^n} + \cdots = \frac{\frac{3}{5^3}}{1 - \frac{1}{5}} = \frac{3}{100}. \quad \Box$$

(d) The series is the sum of two convergent geometric series, so it converges. First,

$$\sum_{n=2}^{\infty} \frac{6^n}{7^n} = \frac{\frac{6^2}{7^2}}{1 - \frac{6}{7}} = \frac{36}{7}.$$

Next,

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{4^n} = \frac{\frac{1}{4^2}}{1 - \left(-\frac{1}{4}\right)} = \frac{1}{20}.$$

Hence,

$$\sum_{n=2}^{\infty} \left(\frac{6^n}{7^n} + 2 \cdot \frac{(-1)^n}{4^n}\right) = \frac{36}{7} + 2 \cdot \frac{1}{20} = \frac{367}{70}. \quad \Box$$

(e) The series $\displaystyle\sum_{n=3}^{\infty} \frac{4^n}{5^n}$ is a convergent geometric series, but $\displaystyle\sum_{n=3}^{\infty} \frac{5^n}{4^n}$ is a divergent geometric series, since the ratio $\dfrac{5}{4}$ is greater than 1. Hence, the given series diverges — in fact, it diverges to $+\infty$. $\Box$

(f) Note that

$$\sum_{n=3}^{\infty} \ln \frac{n}{n+1} = \sum_{n=3}^{\infty} (\ln n - \ln(n+1)).$$

Writing out the first few terms, you can see that the series converges by telescoping:

$$\sum_{n=3}^{\infty} (\ln n - \ln(n+1)) = (\ln 3 - \ln 4) + (\ln 4 - \ln 5) + (\ln 5 - \ln 6) + \cdots = \ln 3. \quad \Box$$

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20. (a) Find the partial fractions decomposition of $\dfrac{2}{(2k+1)(2k+3)}$.

(b) Use (a) to find the sum of the series

$$\sum_{k=1}^{\infty} \frac{2}{(2k+1)(2k+3)}.$$

(a)

$$\frac{2}{(2k+1)(2k+3)} = \frac{A}{2k+1} + \frac{B}{2k+3}$$
$$2 = A(2k+3) + B(2k+1)$$

Set $x = -\dfrac{1}{2}$: I get $2 = 2A$, so $A = 1$.

Set $x = -\frac{3}{2}$: I get $2 = -2B$, so $B = -1$.
Therefore,

$$\frac{2}{(2k+1)(2k+3)} = \frac{1}{2k+1} - \frac{1}{2k+3}. \quad \square$$

(b)

$$\sum_{k=1}^{\infty} \frac{2}{(2k+1)(2k+3)} = \sum_{k=1}^{\infty} \left(\frac{1}{2k+1} - \frac{1}{2k+3}\right) = \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{9}\right) + \cdots.$$

The second fraction in each pair cancels with the first fraction in the next pair. The only one that isn't cancelled is the very first one: $\frac{1}{3}$. Therefore,

$$\sum_{k=1}^{\infty} \frac{2}{(2k+1)(2k+3)} = \frac{1}{3}. \quad \square$$

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21. Find series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ such that both series diverge, and:

(a) $\sum_{n=1}^{\infty} (a_n + b_n)$ diverges.

(b) $\sum_{n=1}^{\infty} (a_n + b_n)$ converges.

(a) Let $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n}$. Then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ both diverge, because they're harmonic.

And $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} \frac{2}{n}$ diverges as well, since it's twice the harmonic series. $\square$

(b) Let $a_n = \frac{1}{n}$ and $b_n = -\frac{1}{n}$. Then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges because it's the harmonic series, and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} -\frac{1}{n}$ diverges because it's the negative of the harmonic series.

However, $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} 0$ converges, and its sum is 0. $\square$

This problem shows that the term-by-term sum of two divergent series can either converge or diverge.

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22. Calvin Butterball notes that $\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$, and concludes that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ converges by the Zero Limit Test. What's wrong with his reasoning?

The Zero Limit Test says that if the limit of the terms **is not** 0, then the series diverges. It does **not** say that if the limit of the terms **is** 0, then the series converges. (The second statement is called the **converse** of the first; the converse of a statement is not the same as, or equivalent to, the statement.)

In fact, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, because it's a p-series with $p = \frac{1}{2} < 1$. $\square$

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23. If the series $\sum_{k=17}^{\infty} a_k$, converges, does the series $\sum_{k=1}^{\infty} a_k$ converge?

If the series $\sum_{k=17}^{\infty} a_k$ converges, then the series $\sum_{k=1}^{\infty} a_k$ converges. They only differ in the first 16 terms, and a finite number of terms cannot affect the convergence or divergence of an infinite series. $\square$

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24. Does the series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{2k+1}{4k+3}$ converge?

The series alternates, but

$$\lim_{k \to \infty} \frac{2k+1}{4k+3} = \frac{1}{2}.$$

The $(-1)^{k+1}$ causes the terms to oscillate in sign, so

$$\lim_{k \to \infty} (-1)^{k+1} \frac{2k+1}{4k+3} \quad \text{is undefined.}$$

The series diverges by the Zero Limit Test. $\square$

---

25. Determine whether the series converges or diverges: $\sum_{k=1}^{\infty} \frac{3^k + 2^k}{6^k}$.

The series is the sum of two convergent geometric series; in fact, its sum is

$$\sum_{k=1}^{\infty} \frac{3^k + 2^k}{6^k} = \sum_{k=1}^{\infty} \frac{3^k}{6^k} + \sum_{k=1}^{\infty} \frac{2^k}{6^k} = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k + \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^k = \frac{\frac{1}{2}}{1 - \frac{1}{2}} + \frac{\frac{1}{3}}{1 - \frac{1}{3}} = 1 + \frac{1}{2} = \frac{3}{2}. \quad \square$$

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26. Determine whether the series converges or diverges: $\sum_{k=3}^{\infty} \frac{k^2 - 3k + 2}{k^4}$.

$$\sum_{k=3}^{\infty} \frac{k^2 - 3k + 2}{k^4} = \sum_{k=3}^{\infty} \frac{1}{k^2} - 3 \sum_{k=3}^{\infty} \frac{1}{k^3} + 2 \sum_{k=3}^{\infty} \frac{1}{k^4}.$$

The series on the right are convergent p-series. Hence, the original series converges. $\square$

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27. Determine whether the series converges or diverges: $\sum_{k=1}^{\infty} \sqrt{\tan^{-1} k}$.

Note that

$$\lim_{k \to \infty} \sqrt{\tan^{-1} k} = \sqrt{\frac{\pi}{2}} \neq 0.$$

Hence, the series diverges by the Zero Limit Test. □

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28. Determine whether the series converges or diverges: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{1.05}}$.

Since $1.05 > 1$, the series is a convergent p-series. □

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29. Determine whether the series converges or diverges: $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$.

Let $f(x) = \dfrac{1}{x(\ln x)^2}$. It is positive and continuous for $x \geq 2$. The derivative is

$$f'(x) = \frac{-2}{x^2(\ln x)^3} - \frac{1}{x^2(\ln x)^2}.$$

$f'(x) < 0$ for $x \geq 2$, so f decreases for $x \geq 2$. The hypotheses of the Integral Test are satisfied. Compute the integral:

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\, dx = \lim_{p \to \infty} \left[-\frac{1}{\ln x}\right]_2^p = \lim_{p \to \infty} \left(-\frac{1}{\ln p} + \frac{1}{\ln 2}\right) = \frac{1}{\ln 2}.$$

Since the integral converges, the series converges by the Integral Test. □

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30. Determine whether the series converges or diverges: $\displaystyle\sum_{k=1}^{\infty} \frac{2}{3k+5}$.

Let $f(x) = \dfrac{2}{3x+5}$. Then f is positive and continuous for $x \geq 1$. The derivative is

$$f'(x) = \frac{-6}{(3x+5)^2}.$$

$f'(x) < 0$ for $x \geq 1$, so f decreases for $x \geq 1$. The hypotheses of the Integral Test are satisfied. Compute the integral:

$$\int_1^{\infty} \frac{2}{3x+5}\, dx = \lim_{p \to \infty} \left[\frac{2}{3} \ln|3x+5|\right]_1^p = \frac{2}{3} \lim_{p \to \infty} (\ln|3p+5| - \ln 8) = +\infty.$$

The limit diverges, so the integral diverges. Therefore, the series diverges, by the Integral Test. □

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31. Determine whether the series converges or diverges: $\displaystyle\sum_{k=1}^{\infty} \frac{e^x}{e^{2x}+1}$.

The terms are positive, and $f(x) = \dfrac{e^x}{e^{2x}+1}$ is continuous for all x.

$$f'(x) = \frac{(e^{2x}+1)(e^x) - (e^x)(2e^{2x})}{(e^{2x}+1)^2} = \frac{e^x - e^{3x}}{(e^{2x}+1)^2} = \frac{e^x(1-e^{2x})}{(e^{2x}+1)^2} < 0 \quad \text{for} \quad x \geq 1.$$

Hence, the terms decrease.

$$\int_1^{\infty} \frac{e^x}{e^{2x}+1}\, dx = \lim_{b \to \infty} \int_1^b \frac{e^x}{e^{2x}+1}\, dx = \lim_{b \to \infty} \int_e^{e^b} \frac{1}{u^2+1}\, du =$$

$$\left[u = e^x, \quad du = e^x\,dx, \quad dx = \frac{du}{e^x}; \quad x = 1, \quad u = e; \quad x = b, \quad u = e^b\right]$$

$$\lim_{b\to\infty} \left[\tan^{-1} u\right]_e^{e^b} = \lim_{b\to\infty} \left(\tan^{-1} e^b - \tan^{-1} e\right) = \frac{\pi}{2} - \tan^{-1} e.$$

The integral converges, so the series converges by the Integral Test. □

He who has overcome his fears will truly be free. - ARISTOTLE