Math 311
9-29-2020

Review Sheet for the Final

These problems are provided to help you study. The presence of a problem on this handout does not imply that there *will* be a similar problem on the test. And the absence of a topic does not imply that it *won't* appear on the test.

1. Multiply the matrices:

$$\begin{bmatrix} -1 & -1 & 0 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 \\ -5 & -1 \\ 3 & -5 \end{bmatrix}$$

2. Combine the matrices:

$$\begin{bmatrix} -3 & 1 & 1 \\ -5 & -5 & 2 \end{bmatrix} \begin{bmatrix} 6 & 4 \\ 1 & -4 \\ -5 & -6 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 3 & 4 \end{bmatrix}$$

3. Suppose

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 6.$$

Find the values of the following determinants.

(a) $\det \begin{bmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{bmatrix}$

(b) $\det \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}$

(c) $\det \begin{bmatrix} a & b & c \\ a+d & b+e & c+f \\ g & h & i \end{bmatrix}$

4. Find a vector:

(a) Going from the point $P(2, -1, 9)$ to the point $Q(3, 3, -7)$.

(b) Which has length 5 and the same direction as the vector $(2, 2, -1)$.

(c) Which is nonzero and is perpendicular to the vector $(2, -5)$.

5. Find the cosine of the angle between the vectors $(1, 2, 4)$ and $(-2, 5, -3)$.

6. Find two unit vectors perpendicular to both $(1, 1, -2)$ and $(0, 4, 5)$.

7. Give an example of three nonzero vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ such that $\vec{a} \cdot \vec{b} = \vec{a}\ cdot\vec{c}$, but $\vec{b} \neq \vec{c}$.

8. Find the area of the triangle with vertices $A(2, 1, 1)$, $B(-3, 0, 1)$, and $C(5, 0, 4)$.

9. Find the parametric and the symmetric equations for the line which passes through the point $(6, -2, 5)$ and is parallel to the vector $(3, -8, 13)$.

10. Find the parametric and symmetric equations of the line which passes through the points $P(3, 4, 6)$ and $Q(-1, 3, 2)$.

11. Determine whether the following lines are parallel, skew, or intersect. If they intersect, find the point of intersection.

$$x = 2 + t, \quad y = 3 - t, \quad z = 4 + 2t,$$
$$x = 1 + s, \quad y = 6 - 2s, \quad z = 3s.$$

12. Determine whether the lines are parallel, skew, or intersecting:

$$x = 2 + t, \quad y = 3 - t, \quad z = 4 + 2t,$$
$$x = 2 + 2s, \quad y = 4 - s, \quad z = 3s.$$

13. Find the distance between the planes

$$x + 2y - 5z = 0 \quad \text{and} \quad x + 2y - 5z = 8.$$

14. Show that the following lines are skew, and find the distance between them.

$$x = 2 - t, \quad y = 3 + 4t, \quad z = 2t,$$
$$x = -1 + u, \quad y = 2, \quad z = -1 + 2u.$$

15. Show that the following lines are parallel, and find the distance between them.

$$x = t, \quad y = 1 + t, \quad z = 1 - t,$$
$$x = 1 - 2s, \quad y = 1 - 2s, \quad z = 2s.$$

16. Find the distance from the point $P(1, 2, 1)$ to the plane $x - 2y - 4z = 8$.

17. (a) Show that the following lines are parallel:

$$x = 1 + 2t, \quad y = 1 - t, \quad z = 3t$$
$$x = -4s, \quad y = 3 + 2s, \quad z = 5 - 6s$$

(b) Find an equation for the plane which contains the lines.

(c) Find the distance between the lines.

18. Find the point of intersection of the line

$$x = 3 + t, \quad y = 5 + 2t, \quad z = 2 - 2t \quad \text{and the plane} \quad 2x + y - z = 3.$$

19. Find the equation of the plane containing the points $P(4, -3, 1)$, $Q(6, -4, 7)$, and $R(1, 2, 2)$.

20. Find the (natural) domain of $f(x, y) = \dfrac{x}{(x - 2)(y - 5)}$.

21. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + 3xy + y^2}{x^2 + y^2}$ is undefined.

22. Let $f(x, y, z) = x^2y^2 - 2xyz + z^2$.

(a) Find the rate of most rapid increase at $(1, -1, 1)$.

(b) Find a unit vector pointing in the direction of most rapid increase.

23. Find the rate of change of $f(x,y,z) = (x+2y+3z)^2 + 3x - 5y + z$ at $(1,-2,1)$ in the direction of the point $(-2,2,13)$.

24. Find the rate of change of $f(x,y) = \dfrac{x^2}{y^2} + 5x^2 - xy$ at the point $(1,1)$ in the direction:

(a) Given by the vector $\vec{v} = (3,-4)$.

(b) Toward the point $(9,-14)$.

25. Construct the Taylor series at $(2,1)$ through terms of the second order for

$$f(x,y) = x^3y + 2xy + \frac{1}{y}.$$

26. Find the equations of the tangent plane and the normal line to the surface

$$x = uv, \quad y = u^2 + v^2, \quad z = u^2 - v^2, \quad \text{at the point} \quad (u,v) = (2,1).$$

27. Suppose

$$(x,y) = (u^3 + v^3, 4uv), \quad (u,v) = (\cos s + \sin t, \sin s - \cos t).$$

Find $\dfrac{\partial x}{\partial s}$ and $\dfrac{\partial y}{\partial t}$.

28. Suppose $w = f(x,y,z)$, $x = p(u,v)$, $y = q(u,v)$, and $z = r(u,v)$.

(a) Use the Chain Rule to find an expression for $\dfrac{\partial w}{\partial u}$.

(b) Use the Chain Rule to find an expression for $\dfrac{\partial^2 w}{\partial u^2}$.

29. (a) Parametrize the surface $x^2 + 9y^2 + z^2 = 16$.

(b) The vertices of a parallelogram, listed counterclockwise, are $A(3,4,-9)$, $B(5,9,-14)$, $C(12,8,-14)$, and $D(10,3,-9)$. Parametrize the parallelogram.

(c) Parametrize the surface generated by revolving the curve $y = x^3 - 2$ about the y-axis.

(d) Parametrize the surface $x^2 - 4y^2 - z^2 = 16$.

30. Locate and classify the critical points of

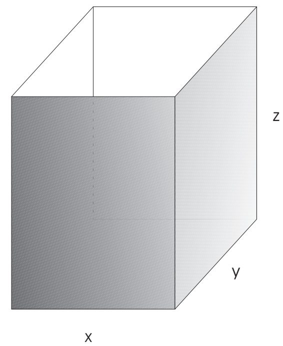
$$z = 2x^3 - 3x^2y + \frac{4}{3}y^3 - 4y + 6.$$

31. Locate and classify the critical points of

$$z = \frac{1}{2}x^2y - 2xy + 2x^2 - 8x + \frac{3}{4}y^2.$$

Show your work!

32. Find the dimensions of the rectangular box with no top having maximal volume and surface area 48.



33. (a) Parametrize the segment from $(1, 4, -9)$ to $(2, 3, 1)$.

(b) Parametrize the curve of intersection of the cylinder $x^2 + y^2 = 25$ and the plane $z = 4x - 2y + 3$.

34. The acceleration function for a cheesesteak sub moving in space is

$$\vec{a}(t) = \left(12t^2 + 4, \frac{8}{t^3}, 0\right).$$

Find the position function $\vec{r}(t)$, given that

$$\vec{r}(1) = (4, 5, 3) \quad \text{and} \quad \vec{v}(1) = (8, -4, 3).$$

35. Find the unit tangent vector to the curve

$$\vec{r}(t) = (2\tan^{-1} t, e^{t^2}, t^2 - t + 1) \quad \text{at} \quad t = 1.$$

36. Find the curvature of $\vec{\sigma}(t) = (t^2 + t + 1, t^2 - t + 1, t^3 + 1)$ at $t = 1$.

37. For the curve $y = x^3 + 5x + 1$, find the unit tangent at $x = 1$, the unit normal at $x = 1$, the curvature at $x = 1$, and an equation for the osculating circle at $x = 1$.

38. Find the unit tangent, the unit normal, the curvature, and the equation of the osculating circle for the curve

$$\vec{\sigma}(t) = ((t + 1)^2, t^3 + 2t + 1), \quad \text{at the point} \quad t = 1.$$

39. Find the volume of the region in the first octant cut off by the plane $2x + y + 2z = 8$.

40. Compute the volume of the solid bounded below by $z = 0$, above by $z = \ln(1 + x^2 + y^2)$, and lying above the region

$$\left\{ \begin{array}{c} 0 \le x \le a \\ 0 \le y \le \sqrt{a^2 - x^2} \end{array} \right\}$$

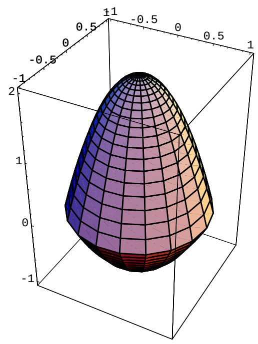
41. Compute $\displaystyle\int_0^1 \int_{-\sqrt{1-y}}^{\sqrt{1-y}} e^{3x - x^3}\, dx\, dy$.

42. Compute $\displaystyle\iiint_R (6x + 4y)\, dV$, where R is the region in the first octant bounded above by $z = y^2$ and bounded on the side by $x + y = 1$.

43. Compute

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{1+(x^2+y^2+z^2)^{3/2}}\,dz\,dy\,dx.$$

44. The solid bounded above by $z = 2 - 2x^2 - 2y^2$ and below by $z = x^2 + y^2 - 1$ has density $\rho = 2$. Find the mass and the center of mass.



45. (a) Parametrize the surface generated by revolving $y = x^2$ for $0 \le x \le 1$, about the x-axis.

(b) Find the area of the surface.

You may want to make use of the following formula:

$$\int u^2\sqrt{a^2u^2+1}\,du = \frac{1}{a^3}\left(\frac{au}{4}(a^2u^2+1)^{3/2} - \frac{au}{8}\sqrt{a^2u^2+1} - \frac{1}{8}\ln|\sqrt{a^2u^2+1}+au|\right) + C.$$

46. A wire is made of the three segments connecting the points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. The density of the wire is $\delta = x + y + z$. Find its mass.

47. Let S be the triangle with vertices $P(1,1,2)$, $Q(2,3,1)$, and $R(-1,2,0)$.
Compute

$$\iint_S (x+4y+z)\,dS.$$

48. Compute $\int_{\vec{\sigma}} f\,ds$, where $f(x,y) = x^2 - y^2$ and $\vec{\sigma}(t) = (e^t\cos 3t, e^t \sin 3t)$, $-1 \le t \le 1$.

49. Let

$$\vec{\sigma}(t) = \left(te^{t-1}, t^3, \sin\left(\frac{\pi t}{2}\right)\right), \quad 0 \le t \le 1.$$

Compute

$$\int_{\vec{\sigma}} (y - z^2)\,dx + (x - 2y + 2yz)\,dy + (y^2 + 2z - 2xz)\,dz.$$

50. Let

$$\vec{F} = \left(\frac{x}{x^2+y^2+z^2+1}, \frac{y}{x^2+y^2+z^2+1}, \frac{z}{x^2+y^2+z^2+1}\right).$$

Let $\vec{\sigma}(t)$ be *any* path from *any* point on the sphere $x^2+y^2+z^2 = 1$ to *any* point on the sphere $x^2+y^2+z^2 = 5$. Compute $\int_{\vec{\sigma}} \vec{F}\cdot\vec{ds}$.

51. (a) Let $\vec{\sigma}$ denote the circle $x^2 + y^2 = 2x$, traversed counterclockwise. Compute

$$\int_{\vec{\sigma}} (2xy - x^2)\,dx + (xy + x^2)\,dy.$$

(b) The vector field in the integral is not conservative, but the integral around the closed curve $\vec{\sigma}$ is 0. Is there anything wrong with this?

52. Consider the ellipse

$$x = a\cos t, \quad y = b\sin t, \qquad 0 \le t \le 2\pi.$$

Use Green's Theorem to show that the area of the ellipse is πab.

53. Let $\vec{\sigma}$ be the path which starts at $(2, 0)$, goes around the circle $x^2 + y^2 = 4$ in the counterclockwise direction, traverses the segment from $(2, 0)$ to $(1, 0)$, goes around the circle $x^2 + y^2 = 1$ in the clockwise direction, and traverses the segment from $(1, 0)$ to $(2, 0)$. Compute $\displaystyle\int_{\vec{\sigma}} -y\,dx + x\,dy$.

54. Compute the circulation of $\vec{F} = (yz, xz, -xy)$ counterclockwise (as viewed from above) around the triangle with vertices $A(1, 2, 1)$, $B(2, 1, 4)$, and $C(-3, 1, 1)$.

55. Let $\vec{\sigma}$ be the curve of intersection of the plane $z = x$ and the cylinder $x^2 + y^2 = 1$, traversed counterclockwise as viewed from above. Compute the circulation of $\vec{F} = (x^2y^3, 1, z)$ around $\vec{\sigma}$:

(a) Directly, by parametrizing the curve and computing the line integral.

(b) Using Stokes' theorem.

56. Let R be the solid region in the first octant cut off by the sphere $x^2 + y^2 + z^2 = 1$. Compute the flux out through the boundary of R of the vector field

$$\vec{F} = \left(x^3 + yz, y^3 - xz, z^3 + 2xy\right).$$

Solutions to the Review Sheet for the Final

1. Multiply the matrices:

$$\begin{bmatrix} -1 & -1 & 0 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 \\ -5 & -1 \\ 3 & -5 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 0 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 \\ -5 & -1 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -15 & -5 \end{bmatrix} \quad \square$$

---

2. Combine the matrices:

$$\begin{bmatrix} -3 & 1 & 1 \\ -5 & -5 & 2 \end{bmatrix} \begin{bmatrix} 6 & 4 \\ 1 & -4 \\ -5 & -6 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & 1 \\ -5 & -5 & 2 \end{bmatrix} \begin{bmatrix} 6 & 4 \\ 1 & -4 \\ -5 & -6 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -22 & -22 \\ -45 & -12 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -22 & -24 \\ -42 & -8 \end{bmatrix} \quad \square$$

---

3. Suppose

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 6.$$

Find the values of the following determinants.

(a) $\det \begin{bmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{bmatrix}$

(b) $\det \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}$

(c) $\det \begin{bmatrix} a & b & c \\ a+d & b+e & c+f \\ g & h & i \end{bmatrix}$

(a)

$$\det \begin{bmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{bmatrix} = 3 \cdot \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 18. \quad \square$$

(b)

$$\det \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix} = -\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = -6. \quad \square$$

(c) Adding a row to another row does not change the determinant. So

$$\det \begin{bmatrix} a & b & c \\ a+d & b+e & c+f \\ g & h & i \end{bmatrix} = \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 6. \quad \square$$

---

4. Find a vector:

(a) Going from the point $P(2,-1,9)$ to the point $Q(3,3,-7)$.

(b) Which has length 5 and the same direction as the vector $(2,2,-1)$.

(c) Which is nonzero and is perpendicular to the vector $(2,-5)$.

(a)

$$\overrightarrow{PQ} = (3-2, 3-(-1), -7-9) = (1,4,-16). \quad \square$$

(b) $\|(2,2,-1)\| = \sqrt{4+4+1} = 3$, so the vector $\dfrac{1}{3}(2,2,-1)$ is a vector of length 1 having the same direction as $(2,2,-1)$.

Multiplying a vector by 5 multiplies its length by 5 without changing its direction. So $\dfrac{5}{3}(2,2,-1)$ is a vector of length 5 having the same direction as $(2,2,-1)$. $\square$

(c) Suppose (a,b) is perpendicular to $(2,-5)$. Then the dot product of the two vectors is 0:

$$(a,b)\cdot(2,-5) = 0, \quad \text{or} \quad 2a - 5b = 0.$$

This equation has infinitely many solutions, and any nonzero solution is a correct answer to the question. For example, if I set $a = 10$, then $b = 4$, so $(10, 4)$ is a nonzero vector which is perpendicular to $(2, -5)$. □

---

5. Find the cosine of the angle between the vectors $(1, 2, 4)$ and $(-2, 5, -3)$.

$$\cos\theta = \frac{(1,2,4)\cdot(-2,5,-3)}{\|(1,2,4)\|\|(-2,5,-3)\|} = \frac{-2+10-12}{\sqrt{21}\sqrt{38}} = -\frac{4}{\sqrt{798}} = -0.14159\ldots. \quad \square$$

---

6. Find two unit vectors perpendicular to both $(1, 1, -2)$ and $(0, 4, 5)$.

The cross product of the two vectors is perpendicular to the two vectors:

$$(1,1,-2)\times(0,4,5) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & -2 \\ 0 & 4 & 5 \end{vmatrix} = (13, -5, 4).$$

The length is $\|(13, -5, 4)\| = \sqrt{210}$. Therefore, $\pm\frac{1}{\sqrt{210}}(13, -5, 4)$ are two unit vectors perpendicular to both $(1, 1, -2)$ and $(0, 4, 5)$. □

---

7. Give an example of three nonzero vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ such that $\vec{a}\cdot\vec{b} = \vec{a}\ cdot \vec{c}$, but $\vec{b} \neq \vec{c}$.

There are many possibilities; I made up an example by picking some numbers at random and adjusting them to make things work. For instance, if $\vec{a} = (1, 3)$, $\vec{b} = (2, -4)$, and $\vec{c} = (-1, -3)$, then

$$\vec{a}\cdot\vec{b} = (1,3)\cdot(2,-4) = -10 \quad \text{and} \quad \vec{a}\cdot\vec{c} = (1,3)\cdot(-1,-3) = -10.$$

However, $\vec{b} \neq \vec{c}$.

This example shows that in a dot product equation like $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$, it wouldn't be legal to "cancel" the $\vec{a}$'s to get $\vec{b} = \vec{c}$. The dot product doesn't behave in every like like multiplication of numbers. □

---

8. Find the area of the triangle with vertices $A(2, 1, 1)$, $B(-3, 0, 1)$, and $C(5, 0, 4)$.

$$\overrightarrow{AB} = (-5, -1, 0), \quad \overrightarrow{AC} = (3, -1, 3),$$

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -5 & -1 & 0 \\ 3 & -1 & 3 \end{vmatrix} = (-3, 15, 8).$$

The length of $\overrightarrow{AB}\times\overrightarrow{AC}$ is the area of the parallelogram determined by $\overrightarrow{AB}$ and $\overrightarrow{AC}$.
The area of the triangle is half the area of the parallelogram:

$$\frac{1}{2}\|(-3, 15, 8)\| = \frac{1}{2}\sqrt{9 + 225 + 64} = \frac{\sqrt{298}}{2} = 8.63133\ldots. \quad \square$$

---

9. Find the parametric and the symmetric equations for the line which passes through the point $(6, -2, 5)$ and is parallel to the vector $(3, -8, 13)$.

The parametric equations are

$$x - 6 = 3t, \quad y + 2 = -8t, \quad z - 5 = 13t.$$

Solve each of these equations for t and equate the results to get the symmetric equations:

$$\frac{x-6}{3} = \frac{y+2}{-8} = \frac{z-5}{13}. \quad \square$$

---

10. Find the parametric and symmetric equations of the line which passes through the points $P(3, 4, 6)$ and $Q(-1, 3, 2)$.

I need a point on the line and a vector parallel to the line.
For the point on the line, I can take either P or Q; I'll use $P(3, 4, 6)$.
Since P and Q are on the line, the vector $\vec{PQ} = (-4, -1, -4)$ is parallel to the line.
Hence, the parametric equations for the line are

$$x - 3 = -4t, \quad y - 4 = -t, \quad z - 6 = -4t.$$

The symmetric equations are

$$\frac{x-3}{-4} = -(y-4) = \frac{z-6}{-4}. \quad \square$$

---

11. Determine whether the following lines are parallel, skew, or intersect. If they intersect, find the point of intersection.

$$x = 2 + t, \quad y = 3 - t, \quad z = 4 + 2t,$$

$$x = 1 + s, \quad y = 6 - 2s, \quad z = 3s.$$

The vector $(1, -1, 2)$ is parallel to the first line. The vector $(1, -2, 3)$ is parallel to the second line. The vectors aren't multiples of one another, so the vectors aren't parallel. Therefore, the lines aren't parallel.

Next, I'll check whether the lines intersect.
Solve the x-equations simultaneously:

$$2 + t = 1 + s, \quad s = 1 + t.$$

Set the y-expressions equal, then plug in $s = 1 + t$ and solve for t:

$$3 - t = 6 - 2s \quad 3 - t = 6 - 2(1 + t), \quad 3 - t = 4 - 2t, \quad t = 1.$$

Therefore, $x = 1 + t = 2$.
Check the values for consistency by plugging them into the z-equations:

$$z = 4 + 2t = 6, \quad z = 3s = 6.$$

The equations are consistent, so the lines intersect. If I plug $t = 1$ into the x-y-z equations, I obtain $x = 3$, $y = 2$, and $z = 6$. The lines intersect at $(3, 2, 6)$. $\square$

---

12. Determine whether the lines are parallel, skew, or intersecting:

$$x = 2 + t, \quad y = 3 - t, \quad z = 4 + 2t,$$

$$x = 2 + 2s, \quad y = 4 - s, \quad z = 3s.$$

The vector $(1, -1, 2)$ is parallel to the first line. The vector $(2, -1, 3)$ is parallel to the second line. The vectors aren't multiples, so they aren't parallel. Therefore, the lines aren't parallel.

Set the x-expressions equal:

$$2 + t = x = 2 + 2s, \quad t = 2s.$$

Set the y-expressions equal, plug in $t = 2s$, and solve for s:

$$3 - t = y = 4 - s, \quad 3 - 2s = 4 - s, \quad -1 = s.$$

Plugging this into $t = 2s$ gives $t = -2$.

Finally, plug $t = -2$ and $s = -1$ into the z-expressions:

$$z = 4 + 2t = 4 - 4 = 0, \quad z = 3s = -3.$$

The z-values don't agree, so the lines don't intersect. The lines are skew. □

---

13. Find the distance between the planes

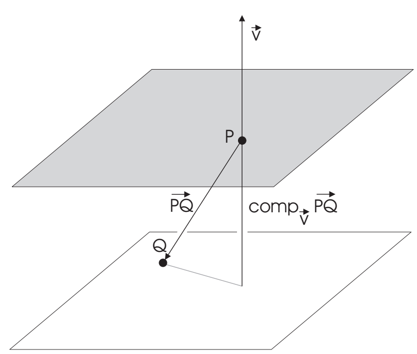
$$x + 2y - 5z = 0 \quad \text{and} \quad x + 2y - 5z = 8.$$

The vector $\vec{v} = (1, 2, -5)$ is perpendicular to both planes, so the planes are parallel.

Set $y = z = 0$ in the first plane equation. This gives $x = 0$. Therefore, the point $P(0, 0, 0)$ is on the first plane.

Set $y = z = 0$ in the second plane equation. This gives $z = 8$. Therefore, the point $Q(8, 0, 0)$ is on the second plane.

Hence, $\overrightarrow{PQ} = (8, 0, 0)$.



The distance is

$$|\operatorname{comp}_{\vec{v}} \overrightarrow{PQ}| = \left| \frac{\overrightarrow{PQ} \cdot \vec{v}}{\|\vec{v}\|} \right| = \left| \frac{(8, 0, 0) \cdot (1, 2, -5)}{\|(1, 2, -5)\|} \right| = \left| \frac{8}{\sqrt{30}} \right| = \frac{8}{\sqrt{30}}. \quad □$$

---

14. Show that the following lines are skew, and find the distance between them.

$$x = 2 - t, \quad y = 3 + 4t, \quad z = 2t,$$

$$x = -1 + u, \quad y = 2, \quad z = -1 + 2u.$$

The vector $(-1,4,2)$ is parallel to the first line, and the vector $(1,0,2)$ is parallel to the second. The vectors are not multiples of each other, so the vectors aren't parallel. Hence, the lines aren't parallel.

If the lines intersect, the distance between them is 0. Hence, I'll just go on to find the distance between the lines. If the distance is nonzero, the lines can't intersect, so they must be skew.

You can think of skew lines as lying in parallel planes. The idea is to find a vector perpendicular to the two lines (or the two planes). Next, find a point P on the first line and a point Q on the second. Finally, the distance will be the absolute value of $\operatorname{comp}_{\vec{v}} \overrightarrow{PQ}$.

I can get a vector $\vec{v}$ perpendicular to both lines by taking the cross product of the vectors parallel to the two lines:

$$(-1,4,2) \times (1,0,2) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 4 & 2 \\ 1 & 0 & 2 \end{vmatrix} = (8,4,-4).$$

Set $t=0$ in the first line to obtain $P(2,3,0)$; set $u=0$ in the second line to obtain $Q(-1,2,-1)$. Then $\overrightarrow{PQ} = (-3,-1,-1)$. Hence,

$$\operatorname{comp}_{\vec{v}} \overrightarrow{PQ} = \frac{(-3,-1,-1)\cdot(8,4,-4)}{\|(8,4,-4)\|} = -\sqrt{6}.$$

The distance is $\sqrt{6} = 2.44948\ldots$. □

---

15. Show that the following lines are parallel, and find the distance between them.

$$x = t, \quad y = 1+t, \quad z = 1-t,$$

$$x = 1-2s, \quad y = 1-2s, \quad z = 2s.$$

The vector $(1,1,-1)$ is parallel to the first line; the vector $(-2,-2,2)$ is parallel to the second line. The second vector is -2 times the first, so the vectors are parallel. Hence, the lines are parallel.

Next, I'll find the distance between the lines. If I set $t=0$, I find that $P(0,1,1)$ lies on the first line; likewise, setting $s=0$, I find that $Q(1,1,0)$ lies on the second line. Now $\overrightarrow{PQ} = (1,0,-1)$; projecting this onto the first line's vector $\vec{v} = (1,1,-1)$, I obtain

$$\operatorname{comp}_{\vec{v}} \overrightarrow{PQ} = \frac{(1,0,-1)\cdot(1,1,-1)}{\|(1,1,-1)\|} = \frac{2}{\sqrt{3}}.$$

I find the distance between the lines using Pythagoras' theorem:

$$\text{distance} = \left[\|\vec{PQ}\|^2 - \left(\operatorname{comp}_{\vec{v}} \overrightarrow{PQ}\right)^2\right]^{1/2} = \sqrt{\frac{2}{3}} = 0.81649\ldots. \quad □$$

---

16. Find the distance from the point $P(1,2,1)$ to the plane $x - 2y - 4z = 8$.

Setting $y=0$ and $z=0$ in the plane equation gives $x=8$. Thus, $Q(8,0,0)$ is a point on the plane, and $\overrightarrow{PQ} = (7,-2,-1)$.

The vector $\vec{v} = (1,-2,-4)$ is perpendicular to the plane.

Now

$$\operatorname{comp}_{\vec{v}} \overrightarrow{PQ} = \frac{\overrightarrow{PQ}\cdot\vec{v}}{\|\vec{v}\|} = \frac{(7,-2,-1)\cdot(1,-2,-4)}{\|(1,-2,-4)\|} = \frac{15}{\sqrt{21}}.$$

The distance from the point to the plane is $\dfrac{15}{\sqrt{21}}$. □

17. (a) Show that the following lines are parallel:

$$x = 1 + 2t, \quad y = 1 - t, \quad z = 3t$$

$$x = -4s, \quad y = 3 + 2s, \quad z = 5 - 6s$$

(b) Find an equation for the plane which contains the lines.

(c) Find the distance between the lines.

(a) The vector $(2, -1, 3)$ is parallel to the first line.
The vector $(-4, 2, -6)$ is parallel to the second line.
Since $(-2)(2, -1, 3) = (-4, 2, -6)$, the vectors are parallel. Therefore, the lines are parallel. $\square$

(b) The vector $(2, -1, 3)$ is parallel to the first line, so I can regard it as lying in the plane.
Setting $t = 0$ produces the point $P(1, 1, 0)$ on the first line.
Setting $s = 0$ produces the point $Q(0, 3, 5)$ on the second line.
The vector $\overrightarrow{PQ} = (-1, 2, 5)$ lies in the plane containing the lines.
The cross product is perpendicular to the plane:

$$(2, -1, 3) \times (-1, 2, 5) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 3 \\ -1 & 2 & 5 \end{vmatrix} = (-11, -13, 3).$$

The point $P(1, 1, 0)$ is on the plane, so the plane is

$$-11(x - 1) - 13(y - 1) + 3(z - 0) = 0, \quad \text{or} \quad -11x - 13y + 3z = -24. \quad \square$$

(c) $\overrightarrow{PQ} = (-1, 2, 5)$ goes from the first line to the second, and $\vec{v} = (2, -1, 3)$ is parallel to the lines.

$$\operatorname{comp}_{\vec{v}} \overrightarrow{PQ} = \frac{(-1, 2, 5) \cdot (2, -1, 3)}{\|(2, -1, 3)\|} = \frac{11}{\sqrt{14}}.$$

The distance is

$$\sqrt{\|\overrightarrow{PQ}\|^2 - \left(\operatorname{comp}_{\vec{v}} \overrightarrow{PQ}\right)^2} = \sqrt{30 - \frac{121}{14}} = \sqrt{\frac{299}{14}} = 1.23511 \ldots. \quad \square$$

18. Find the point of intersection of the line

$$x = 3 + t, \quad y = 5 + 2t, \quad z = 2 - 2t \quad \text{and the plane} \quad 2x + y - z = 3.$$

Plug the expressions for x, y, and z from the line into the equation of the plane:

$$2(3 + t) + (5 + 2t) - (2 - 2t) = 3, \quad 6t = -6, \quad t = -1.$$

Plugging $t = -1$ into the equations for x, y, and z gives $x = 2$, $y = 3$, and $z = 4$. The point of intersection is $(2, 3, 4)$. $\square$

19. Find the equation of the plane containing the points $P(4, -3, 1)$, $Q(6, -4, 7)$, and $R(1, 2, 2)$.

The vectors $\overrightarrow{PQ} = (2, -1, 6)$ and $\overrightarrow{PR} = (-3, 5, 1)$ lie in the plane, so their cross product is perpendicular to the plane. The cross product is

$$(2, -1, 6) \times (-3, 5, 1) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 6 \\ -3 & 5 & 1 \end{vmatrix} = (-31, -20, 7).$$

Since the point $P(4, -3, 1)$ lies on the plane, the plane is

$$-31(x-4) - 20(y+3) + 7(z-1) = 0, \quad \text{or} \quad -31x - 20y + 7z = -57. \quad \square$$

---

20. Find the (natural) domain of $f(x, y) = \dfrac{x}{(x-2)(y-5)}$.

The values $x = 2$ and $y = 5$ cause division by 0. Hence, the domain consists of all of $\mathbb{R}^2$ except the lines $x = 2$ and $y = 5$. $\square$

---

21. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + 3xy + y^2}{x^2 + y^2}$ is undefined.

Approaching $(0, 0)$ along the x-axis $y = 0$, I have

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + 3xy + y^2}{x^2 + y^2} = \lim_{x\to 0} \frac{x^2}{x^2} = \lim_{x\to 0} 1 = 1.$$

Approaching $(0, 0)$ along the line $y = x$, I have

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + 3xy + y^2}{x^2 + y^2} = \lim_{x\to 0} \frac{x^2 + 3x^2 + x^2}{x^2 + x^2} = \lim_{x\to 0} \frac{5x^2}{2x^2} = \lim_{x\to 0} \frac{5}{2} = \frac{5}{2}.$$

Since the function approaches different values as $(0, 0)$ is approached in different ways, the limit is undefined. $\square$

---

22. Let $f(x, y, z) = x^2y^2 - 2xyz + z^2$.

(a) Find the rate of most rapid increase at $(1, -1, 1)$.

(b) Find a unit vector pointing in the direction of most rapid increase.

(a)

$$\nabla f = (2xy^2 - 2yz, 2x^2y - 2xz, -2xy + 2z), \quad \nabla f(1, -1, 1) = (4, -4, 4).$$

The rate of most rapid increase is

$$\|\nabla f\| = \sqrt{4^2 + (-4)^2 + 4^2} = \sqrt{48} = 4\sqrt{3}. \quad \square$$

(b) The gradient points in the direction of most rapid increase. Therefore, a unit vector pointing in the direction of most rapid increase is given by

$$\frac{\nabla f}{\|\nabla f\|} = \frac{(4, -4, 4)}{4\sqrt{3}} = \frac{1}{\sqrt{3}}(1, -1, 1). \quad \square$$

---

23. Find the rate of change of $f(x,y,z) = (x+2y+3z)^2 + 3x - 5y + z$ at $(1,-2,1)$ in the direction of the point $(-2,2,13)$.

$$\nabla f = (2(x+2y+3z)+3, 4(x+2y+3z)-5, 6(x+2y+3z)+1), \quad \nabla f(1,-2,1) = (3,-5,1).$$

The vector from $(1,-2,1)$ to $(-2,2,13)$ is $\vec{v} = (-3,4,12)$.
Therefore,

$$Df_{\vec{v}}(1,-2,1) = (3,-5,1) \cdot \frac{(-3,4,12)}{\|(-3,4,12)\|} = -\frac{17}{13}. \quad \square$$

---

24. Find the rate of change of $f(x,y) = \dfrac{x^2}{y^2} + 5x^2 - xy$ at the point $(1,1)$ in the direction:

(a) Given by the vector $\vec{v} = (3,-4)$.

(b) Toward the point $(9,-14)$.

(a)

$$\nabla f = \left(\frac{2x}{y^2} + 10x - y, -\frac{2x^2}{y^3} - x\right), \quad \nabla f(1,1) = (11,-3).$$

Therefore,

$$Df_{\vec{v}}(1,1) = \nabla f(1,1) \cdot \frac{(3,-4)}{\|(3,-4)\|} = (11,-3) \cdot \frac{(3,-4)}{5} = 9. \quad \square$$

(b) The vector from $(1,1)$ to $(9,-14)$ is $\vec{w} = (8,-15)$.
So

$$Df_{\vec{w}}(1,1) = \nabla f(1,1) \cdot \frac{(8,-15)}{\|(8,-15)\|} = (11,-3) \cdot \frac{(8,-15)}{17} = \frac{133}{17}. \quad \square$$

---

25. Construct the Taylor series at $(2,1)$ through terms of the second order for

$$f(x,y) = x^3 y + 2xy + \frac{1}{y}.$$

$$\frac{\partial f}{\partial x} = 3x^2 y + 2y, \quad \frac{\partial f}{\partial y} = x^3 + 2x - \frac{1}{y^2}.$$

$$\frac{\partial^2 f}{\partial x^2} = 6xy, \quad \frac{\partial^2 f}{\partial x \partial y} = 3x^2 + 2, \quad \frac{\partial^2 f}{\partial y^2} = \frac{2}{y^3}.$$

Plug in $(2,1)$:

$$f(2,1) = 13, \quad \frac{\partial f}{\partial x}(2,1) = 14, \quad \frac{\partial f}{\partial y}(2,1) = 11.$$

$$\frac{\partial^2 f}{\partial x^2}(2,1) = 12, \quad \frac{\partial^2 f}{\partial x \partial y}(2,1) = 14, \quad \frac{\partial^2 f}{\partial y^2}(2,1) = 2.$$

The series is

$$f(x,y) = 13 + 14(x-2) + 11(y-1) + \frac{1}{2}\left(12(x-2)^2 + 28(x-2)(y-1) + 2(y-1)^2\right) + \cdots. \quad \square$$

26. Find the equations of the tangent plane and the normal line to the surface

$$x = uv, \quad y = u^2 + v^2, \quad z = u^2 - v^2, \quad \text{at the point} \quad (u, v) = (2, 1).$$

The point of tangency is $(x, y, z) = (2, 5, 3)$.

$$\vec{T}_u = (v, 2u, 2u), \quad \vec{T}_v = (u, 2v, -2v).$$

$$\vec{T}_u(2, 1) = (1, 4, 4), \quad \vec{T}_v = (2, 2, -2).$$

A normal vector is given by

$$\vec{T}_u \times \vec{T}_v = (1, 4, 4) \times (2, 2, -2) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 4 & 4 \\ 2 & 2 & -2 \end{vmatrix} = (-16, 10, -6).$$

I can divide out a common factor of -2 and get the vector $(8, -5, 3)$. (This isn't required.)
The tangent plane is

$$8(x - 2) - 5(y - 5) + 3(z - 3) = 0, \quad \text{or} \quad 8x - 5y + 3z = 0.$$

The normal line is

$$x - 2 = 8t, \quad y - 5 = -5t, \quad z - 3 = 3t. \quad \square$$

---

27. Suppose

$$(x, y) = (u^3 + v^3, 4uv), \quad (u, v) = (\cos s + \sin t, \sin s - \cos t).$$

Find $\dfrac{\partial x}{\partial s}$ and $\dfrac{\partial y}{\partial t}$.

$$\frac{\partial x}{\partial s} = \frac{\partial x}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial s} = (3u^2)(-\sin s) + (3v^2)(\cos s).$$

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial t} = (4v)(\cos t) + (4u)(\sin t). \quad \square$$

---

28. Suppose $w = f(x, y, z)$, $x = p(u, v)$, $y = q(u, v)$, and $z = r(u, v)$.

(a) Use the Chain Rule to find an expression for $\dfrac{\partial w}{\partial u}$.

(b) Use the Chain Rule to find an expression for $\dfrac{\partial^2 w}{\partial u^2}$.

(a)

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}. \quad \square$$

(b)

$$\frac{\partial^2 w}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}\right) =$$

$$\frac{\partial w}{\partial x}\frac{\partial^2 x}{\partial u^2}+\frac{\partial x}{\partial u}\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial x}{\partial u}+\frac{\partial^2 w}{\partial y\partial x}\frac{\partial y}{\partial u}+\frac{\partial^2 w}{\partial z\partial x}\frac{\partial z}{\partial u}\right)+\frac{\partial w}{\partial y}\frac{\partial^2 y}{\partial u^2}+\frac{\partial y}{\partial u}\left(\frac{\partial^2 w}{\partial x\partial y}\frac{\partial x}{\partial u}+\frac{\partial^2 w}{\partial y^2}\frac{\partial y}{\partial u}+\frac{\partial^2 w}{\partial z\partial y}\frac{\partial z}{\partial u}\right)+$$

$$\frac{\partial w}{\partial z}\frac{\partial^2 z}{\partial u^2}+\frac{\partial z}{\partial u}\left(\frac{\partial^2 w}{\partial x\partial z}\frac{\partial x}{\partial u}+\frac{\partial^2 w}{\partial y\partial z}\frac{\partial y}{\partial u}\frac{\partial^2 w}{\partial z^2}\frac{\partial z}{\partial u}\right).\quad\square$$

---

29. (a) Parametrize the surface $x^2+9y^2+z^2=16$.

(b) The vertices of a parallelogram, listed counterclockwise, are $A(3,4,-9)$, $B(5,9,-14)$, $C(12,8,-14)$, and $D(10,3,-9)$. Parametrize the parallelogram.

(c) Parametrize the surface generated by revolving the curve $y=x^3-2$ about the y-axis.

(d) Parametrize the surface $x^2-4y^2-z^2=16$.

(a) $x^2+y^2+z^2=1$ can be parametrized by

$$x=\cos u\cos v,\quad y=\sin u\cos v,\quad z=\sin v.$$

$x^2+9y^2+z^2=1$ can be parametrized by

$$x=\cos u\cos v,\quad y=\frac{1}{3}\sin u\cos v,\quad z=\sin v.$$

Hence, $x^2+9y^2+z^2=16$ can be parametrized by

$$x=4\cos u\cos v,\quad y=\frac{4}{3}\sin u\cos v,\quad z=4\sin v.\quad\square$$

(b)

$$\overrightarrow{AB}=(2,5,-5),\quad \overrightarrow{AD}=(7,-1,0).$$

The parametrization is

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2&7\\5&-1\\-5&0\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}+\begin{bmatrix}3\\4\\-9\end{bmatrix}.$$

$$x=2u+7v+3,\quad y=5u-v+4,\quad z=-5u-9.\quad\square$$

(c) The curve may be parametrized by

$$x=u,\quad y=u^3-2.$$

The surface is

$$x=u\cos v,\quad y=u^3-2,\quad z=u\sin v.\quad\square$$

(d) Write the surface as $4y^2+z^2=x^2-16$.
Setting $x=u$, I obtain $4y^2+z^2=u^2-16$.
$y^2+z^2=1$ can be parametrized by $y=\cos v$ and $z=\sin v$.
$4y^2+z^2=1$ can be parametrized by $y=\frac{1}{2}\cos v$ and $z=\sin v$.
$4y^2+z^2=a$ can be parametrized by $y=\frac{\sqrt{a}}{2}\cos v$ and $z=\sqrt{a}\sin v$.
Thus, $4y^2+z^2=u^2-16$ may be parametrized by

$$y=\frac{\sqrt{u^2-16}}{2}\cos v,\quad z=\sqrt{u^2-16}\sin v.$$

The surface is

$$x = u, \quad y = \frac{\sqrt{u^2 - 16}}{2} \cos v, \quad z = \sqrt{u^2 - 16} \sin v.$$

Note: The surface will only be defined for $u \le -4$ and $u \ge 4$. It is a **hyperboloid of two sheets**, and consists of two pieces. ☐

---

30. Locate and classify the critical points of

$$z = 2x^3 - 3x^2 y + \frac{4}{3} y^3 - 4y + 6.$$

$$z_x = 6x^2 - 6xy, \quad z_y = -3x^2 + 4y^2 - 4,$$
$$z_{xx} = 12x - 6y, \quad z_{yy} = 8y, \quad z_{xy} = -6x.$$

Set the first-order partials equal to 0:

(1) $$6x^2 - 6xy = 0, \quad x(x - y) = 0,$$

(2) $$-3x^2 + 4y^2 - 4 = 0.$$

Solve simultaneously:

(1) $x(x - y) = 0$

$\swarrow$ $\searrow$

$x = 0$
(2) $-3x^2 + 4y^2 - 4 = 0$
$4y^2 - 4 = 0$
$y^2 = 1$

$\swarrow$ $\searrow$

$y = 1$, $(0, 1)$

$y = -1$, $(0, -1)$

$x = y$
$-3x^2 + 4y^2 - 4 = 0$
$y^2 - 4 = 0$
$y^2 = 4$

$\swarrow$ $\searrow$

$y = 2$, $x = 2$, $(2, 2)$

$y = -2$, $x = -2$, $(-2, -2)$

Test the critical points:

point	z_{xx}	z_{yy}	z_{xy}	Δ	result
$(0, 1)$	-6	8	0	-48	saddle
$(0, -1)$	6	-8	0	-48	saddle
$(2, 2)$	12	16	-12	48	min
$(-2, -2)$	-12	-16	12	48	max

☐

---

31. Locate and classify the critical points of

$$z = \frac{1}{2} x^2 y - 2xy + 2x^2 - 8x + \frac{3}{4} y^2.$$

Show your work!

$$z_x = xy - 2y + 4x - 8, \quad z_y = \frac{1}{2}x^2 - 2x + \frac{3}{2}y.$$

$$z_{xx} = y + 4, \quad z_{xy} = x - 2, \quad z_{yy} = \frac{3}{2}.$$

Find the critical points:

$$\begin{gathered} xy - 2y + 4x - 8 = 0 \\ y(x-2) + 4(x-2) = 0 \\ (y+4)(x-2) = 0 \end{gathered}$$

$\swarrow \qquad \searrow$

$$\begin{gathered} y + 4 = 0 \\ y = -4 \\ \frac{1}{2}x^2 - 2x + \frac{3}{2}y = 0 \\ \frac{1}{2}x^2 - 2x - 6 = 0 \\ x^2 - 4x - 12 = 0 \\ (x-6)(x+2) = 0 \end{gathered}$$

$\swarrow \qquad \searrow$

$$\begin{gathered} x = 6 \\ (6, -4) \end{gathered} \qquad \begin{gathered} x = -2 \\ (-2, -4) \end{gathered}$$

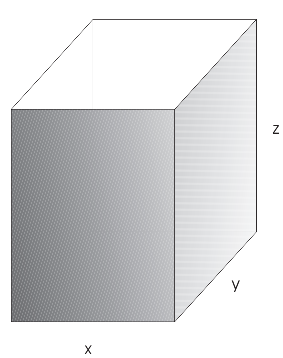
$$\begin{gathered} x - 2 = 0 \\ x = 2 \\ \frac{1}{2}x^2 - 2x + \frac{3}{2}y = 0 \\ -2 + \frac{3}{2}y = 0 \\ y = \frac{4}{3} \\ \left(2, \frac{4}{3}\right) \end{gathered}$$

point	z_{xx}	z_{yy}	z_{xy}	Δ	result
$(6, -4)$	0	$\frac{3}{2}$	4	-16	saddle
$(-2, -4)$	0	$\frac{3}{2}$	-4	-16	saddle
$\left(2, \frac{4}{3}\right)$	$\frac{16}{3}$	$\frac{3}{2}$	0	8	min

$\square$

---

32. Find the dimensions of the rectangular box with no top having maximal volume and surface area 48.



Let x and y be the dimensions of the base, and let z be the height. I want to find the maximum value of $f(x, y) = xyz$ subject to the constraint $48 = xy + 2xz + 2yz$. Write $g(x, y, z) = xy + 2xz + 2yz - 48 = 0$.

I obtain the equations

$$yz = \lambda(y + 2z), \tag{1}$$

$$xz = \lambda(x + 2z), \tag{2}$$

$$xy = \lambda(2x + 2y), \tag{3}$$

$$48 = xy + 2xz + 2yz. \tag{4}$$

Before continuing, note that since x, y, and z are the dimensions of a box, they can't be 0 or negative.

In addition, I may assume that $x, y, z > 0$. For $x = 2$, $y = 2$, and $z = \dfrac{11}{2}$ satisfies the constraint and gives a box of volume 6. So I can certainly do better (in terms of getting a larger volume) than to have one of the dimensions equal 0, which would give a box a volume 0. This implies that I may divide by x, y, or z, and I'll do so below without further comment.

Since $x, y, z \neq 0$, I may assume that $y + 2z \neq 0$. For if $y + 2z = 0$, then the first equation gives $0 = yz$, which would imply that $y = 0$ or $z = 0$. Likewise, I may assume that $x + 2z \neq 0$, and $2x + 2y \neq 0$. This implies that I may divide by $y + 2z$, and I'll do so below without further comment.

Now I'll solve the equations simultaneously.

$$\begin{array}{lc}
(1) & yz = \lambda(y + 2z) \\
 & \lambda = \dfrac{yz}{y + 2z} \\
(2) & xz = \lambda(x + 2z) \\
 & xz = \dfrac{yz(x + 2z)}{y + 2z} \\
 & xz(y + 2z) = yz(x + 2z) \\
 & xyz + 2xz^2 = xyz + 2yz^2 \\
 & 2xz^2 = 2yz^2 \\
 & x = y \\
(3) & xy = \lambda(2x + 2y) \\
 & xy = \dfrac{yz(2x + 2y)}{y + 2z} \\
 & xy(y + 2z) = yz(2x + 2y) \\
 & xy^2 + 2xyz = 2xyz + 2y^2 z \\
 & x = 2z \\
 & z = \dfrac{x}{2} \\
(4) & 48 = xy + 2xz + 2yz \\
 & 48 = x^2 + x^2 + x^2 \\
 & 3x^2 = 48 \\
 & x = 4 \\
 & y = 4 \\
 & z = 2
\end{array}$$

The dimensions $x = 4$, $y = 4$, and $z = 2$ maximize the volume. (I can satisfy the constraint and make the volume arbitrarily small by making one of the dimensions sufficiently small. Thus, the point $(4, 4, 2)$ can't give a min.) □

---

33. (a) Parametrize the segment from $(1, 4, -9)$ to $(2, 3, 1)$.

(b) Parametrize the curve of intersection of the cylinder $x^2 + y^2 = 25$ and the plane $z = 4x - 2y + 3$.

$$(x, y, z) = (1 - t) \cdot (1, 4, -9) + t \cdot (2, 3, 1) = (t + 1, 4 - t, 10t - 9).$$

Hence,

$$x = t + 1, \quad y = 4 - t, \quad z = 10t - 9, \quad 0 \leq t \leq 1. \quad □$$

(b) The circle $x^2 + y^2 = 25$ may be parametrized by

$$x = 5\cos t, \quad y = 5\sin t, \quad 0 \le t \le 2\pi.$$

Plugging these into the plane equation gives

$$z = 4(5\cos t) - 2(5\sin t) + 3 = 20\cos t - 10\sin t + 3.$$

The parametric equations are

$$x = 5\cos t, \quad y = 5\sin t, \quad , z = 20\cos t - 10\sin t + 3, \quad 0 \le t \le 2\pi. \quad \square$$

---

34. The acceleration function for a cheesesteak sub moving in space is

$$\vec{a}(t) = \left(12t^2 + 4, \frac{8}{t^3}, 0\right).$$

Find the position function $\vec{r}(t)$, given that

$$\vec{r}(1) = (4, 5, 3) \quad \text{and} \quad \vec{v}(1) = (8, -4, 3).$$

The acceleration function is the derivative of the velocity function, so the velocity function is the integral of the acceleration function:

$$\vec{v}(t) = \int \left(12t^2 + 4, \frac{8}{t^3}, 0\right)\,dt = \left(4t^3 + 4t, -\frac{4}{t^2}, 0\right) + (c_1, c_2, c_3).$$

To find (c_1, c_2, c_3), I'll plug in the initial condition $\vec{v}(1) = (8, -4, 3)$:

$$\begin{aligned}(8, -4, 3) &= (8, -4, 0) + (c_1, c_2, c_3) \\ (c_1, c_2, c_3) &= (0, 0, 3)\end{aligned}$$

Therefore,

$$\vec{v}(t) = \left(4t^3 + 4t, -\frac{4}{t^2}, 0\right) + (0, 0, 3) = \left(4t^3 + 4t, -\frac{4}{t^2}, 3\right).$$

The velocity function is the derivative of the position function, so the position function is the integral of the velocity function:

$$\vec{r}(t) = \int \left(4t^3 + 4t, -\frac{4}{t^2}, 3\right)\,dt = \left(t^4 + 2t^2, \frac{4}{t}, 3t\right) + (d_1, d_2, d_3).$$

To find (d_1, d_2, d_3), I'll plug in the initial condition $\vec{r}(1) = (4, 5, 3)$:

$$\begin{aligned}(4, 5, 3) &= (3, 4, 3) + (d_1, d_2, d_3) \\ (d_1, d_2, d_3) &= (1, 1, 0)\end{aligned}$$

Therefore,

$$\vec{r}(t) = \left(t^4 + 2t^2, \frac{4}{t}, 3t\right) + (1, 1, 0) = \left(t^4 + 2t^2 + 1, \frac{4}{t} + 1, 3t\right). \quad \square$$

---

35. Find the unit tangent vector to the curve

$$\vec{r}(t) = (2\tan^{-1} t, e^{t^2}, t^2 - t + 1) \quad \text{at} \quad t = 1.$$

$$\vec{r}\,'(t) = \left(\frac{2}{t^2+1}, 2te^{t^2}, 2t - 1\right).$$

Since I have a point to plug in, I'll plug it in now, then compute the length:

$$\vec{r}\,'(1) = (1, 2, 1), \quad \|\vec{r}\,'(1)\| = \sqrt{6}.$$

The unit tangent is

$$\vec{T}(1) = \frac{1}{\sqrt{6}}(1, 2, 1). \quad \square$$

---

36. Find the curvature of $\vec{\sigma}(t) = (t^2 + t + 1, t^2 - t + 1, t^3 + 1)$ at $t = 1$.

$$\vec{\sigma}\,'(t) = (2t + 1, 2t - 1, 3t^2) \quad \text{and} \quad \vec{\sigma}\,''(t) = (2, 2, 6t).$$

So

$$\vec{\sigma}\,'(1) = (3, 1, 3) \quad \text{and} \quad \vec{\sigma}\,''(1) = (2, 2, 6).$$

Then

$$\|\vec{\sigma}\,'(1)\| = \sqrt{9 + 1 + 9} = \sqrt{19},$$

$$\vec{\sigma}\,'(1) \times \vec{\sigma}\,''(1) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 1 & 3 \\ 2 & 2 & 6 \end{vmatrix} = (0, -12, 4), \quad \text{so} \quad \|\vec{\sigma}\,'(1) \times \vec{\sigma}\,''(1)\| = \sqrt{0 + 144 + 16} = \sqrt{160} = 4\sqrt{10}.$$

The curvature is

$$\kappa = \frac{\|\vec{\sigma}\,'(1) \times \vec{\sigma}\,''(1)\|}{\|\vec{\sigma}\,'(1)\|^3} = \frac{4\sqrt{10}}{19\sqrt{19}}. \quad \square$$

---

37. For the curve $y = x^3 + 5x + 1$, find the unit tangent at $x = 1$, the unit normal at $x = 1$, the curvature at $x = 1$, and an equation for the osculating circle at $x = 1$.

Parametrize the curve by

$$x = t, \quad y = t^3 + 5t + 1.$$

Then

$$\vec{r}\,'(t) = (1, 3t^2 + 5), \quad \vec{r}\,'(1) = (1, 8), \quad \|\vec{r}\,'(1)\| = \sqrt{65}.$$

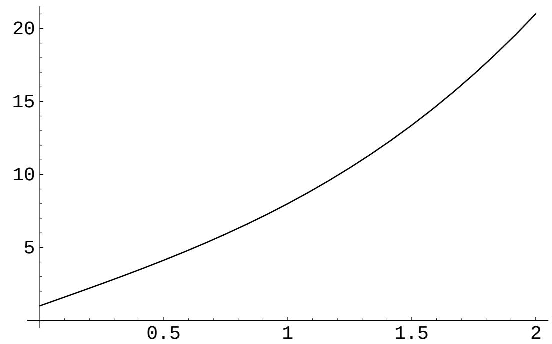
The unit tangent is

$$\vec{T}(1) = \frac{1}{\sqrt{65}}(1, 8).$$

There are two possibilities for the unit normal:

$$\frac{1}{\sqrt{65}}(8, -1) \quad \text{and} \quad \frac{1}{\sqrt{65}}(-8, 1).$$

Here's a picture of the curve near $x = 1$:



The unit normal points up and to the left, so

$$\vec{N}(1) = \frac{1}{\sqrt{65}}(-8, 1).$$

Now

$$f'(x) = 3x^2 + 5, \quad f'(1) = 8,$$
$$f''(x) = 6x, \quad f''(1) = 6.$$

The curvature is

$$\kappa = \frac{6}{(1 + 8^2)^{3/2}} = \frac{6}{65\sqrt{65}}.$$

The radius of curvature is $\dfrac{65\sqrt{65}}{6}$.

When $x = 1$, $y = 6$. The osculating circle is

$$(x, y) = (1, 6) + \frac{65\sqrt{65}}{6} \cdot \frac{1}{\sqrt{65}}(-8, 1) + \frac{65\sqrt{65}}{6} \cdot \frac{1}{\sqrt{65}}(1, 8)\cos t + \frac{65\sqrt{65}}{6} \cdot \frac{1}{\sqrt{65}}(-8, 1)\sin t. \quad \square$$

---

38. Find the unit tangent, the unit normal, the curvature, and the equation of the osculating circle for the curve

$$\vec{\sigma}(t) = ((t + 1)^2, t^3 + 2t + 1), \quad \text{at the point} \quad t = 1.$$
$$\vec{\sigma}\,'(t) = (2(t + 1), 3t^2 + 2), \quad \vec{\sigma}\,'(1) = (4, 5).$$

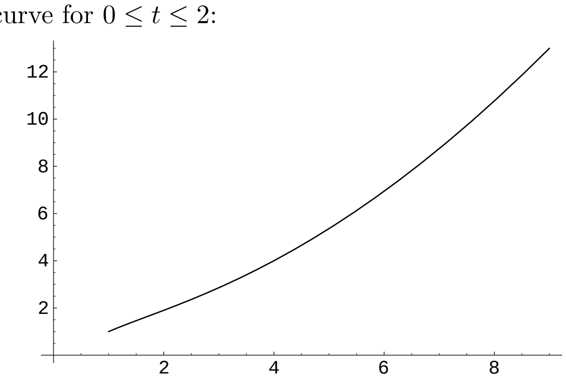
The unit tangent is

$$\vec{T}(1) = \frac{\vec{\sigma}\,'(1)}{\|\vec{\sigma}\,'(1)\|} = \frac{1}{\sqrt{41}}(4, 5).$$

The following unit vectors are clearly perpendicular to $\vec{T}(1)$:

$$\frac{1}{\sqrt{41}}(5, -4) \quad \text{and} \quad \frac{1}{\sqrt{41}}(-5, 4).$$

Here's a picture of the curve for $0 \le t \le 2$:



The unit normal must point upward, so its y-component must be positive. Therefore,

$$\vec{N}(1) = \frac{1}{\sqrt{41}}(-5, 4).$$

For the curvature, I'll use the formula

$$\kappa = \frac{|x'(1)y''(1) - x''(1)y'(1)|}{(x'(1)^2 + y'(1)^2)^{3/2}}.$$

In this case,

$$x' = 2(t+1), \quad x'(1) = 4, \quad y' = 3t^2 + 2, \quad y'(1) = 5,$$
$$x'' = 2, \quad x''(1) = 2, \quad y'' = 6t, \quad y''(1) = 6.$$

Therefore,

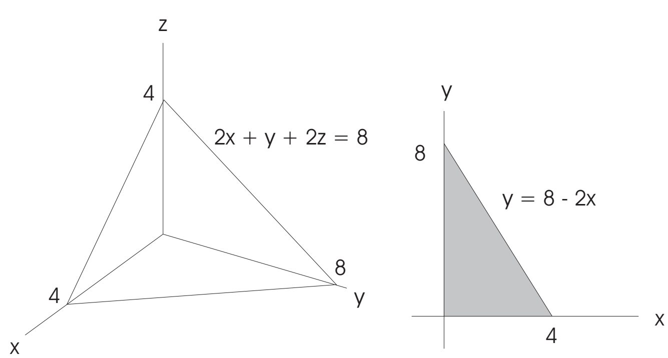
$$\kappa = \frac{|(4)(6) - (2)(5)|}{(4^2 + 5^2)^{3/2}} = \frac{14}{41\sqrt{41}}.$$

The point on the curve is $\vec{\sigma}(1) = (4, 4)$ and the radius of curvature is $R = \dfrac{41\sqrt{41}}{14}$. The osculating circle is

$$(x, y) = (4, 4) + \frac{41\sqrt{41}}{14} \cdot \frac{1}{\sqrt{41}}(-5, 4) + \frac{41\sqrt{41}}{14} \cdot \frac{1}{\sqrt{41}}(4, 5)\cos t + \frac{41\sqrt{41}}{14} \cdot \frac{1}{\sqrt{41}}(-5, 4)\sin t =$$
$$(4, 4) + \frac{41}{14}(-5, 4) + \frac{41}{14}(4, 5)\cos t + \frac{41}{14}(4, 5)\sin t. \quad \square$$

---

39. Find the volume of the region in the first octant cut off by the plane $2x + y + 2z = 8$.



The first picture shows the plane. The projection of the region into the x-y-plane is shown in the second picture. The projection is

$$\left\{ \begin{matrix} 0 \le x \le 4 \\ 0 \le y \le -2x + 8 \end{matrix} \right\}$$

Therefore, the volume is:

$$\int_0^4 \int_0^{-2x+8} \left(4 - x - \frac{1}{2}y\right) dy\, dx = \int_0^4 \left[4y - xy - \frac{1}{4}y^2\right]_0^{-2x+8} dx =$$
$$\int_0^4 (x-4)^2\, dx = \left[\frac{1}{3}(x-4)^3\right]_0^4 = \frac{64}{3}. \quad \square$$

---

40. Compute the volume of the solid bounded below by $z = 0$, above by $z = \ln(1 + x^2 + y^2)$, and lying above the region

$$\left\{ \begin{matrix} 0 \le x \le a \\ 0 \le y \le \sqrt{a^2 - x^2} \end{matrix} \right\}$$

Convert to polar. The volume is

$$\int_0^{\pi/2} \int_0^a r \ln(1 + r^2)\, dr\, d\theta = \frac{\pi}{2} \left[\frac{1}{2} r^2 \ln(r^2 + 1) + \frac{1}{2} \ln(r^2 + 1) - \frac{1}{2} r^2 \right]_0^a = \frac{\pi}{4} \left((a^2 + 1) \ln(a^2 + 1) - a^2 \right).$$

Here's the work for the integral. First, using integration by parts,

$$\int \ln x\, dx = x \ln x - \int \frac{1}{x} \cdot x\, dx = x \ln x - \int dx = x \ln x - x + c.$$

	$\dfrac{d}{dx}$	$\displaystyle\int dx$
$+$	$\ln x$	1
$-$	$\dfrac{1}{x}$	x

Next,

$$\int r \ln(r^2 + 1)\, dr = \int r \ln u \cdot \frac{du}{2r} = \frac{1}{2} \int \ln u\, du =$$

$$\left[u = r^2 + 1, \quad du = 2r\, dr, \quad dr = \frac{du}{2r} \right]$$

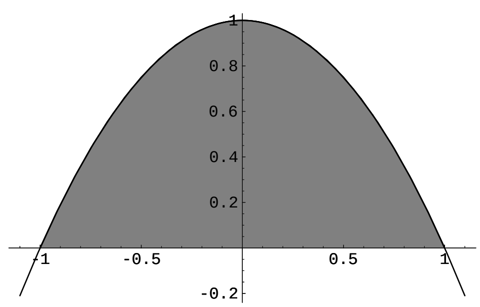
$$\frac{1}{2}(u \ln u - u) + c = \frac{1}{2} \left((r^2 + 1) \ln(r^2 + 1) - (r^2 + 1) \right) + c. \quad \square$$

---

41. Compute $\displaystyle\int_0^1 \int_{-\sqrt{1-y}}^{\sqrt{1-y}} e^{3x - x^3}\, dx\, dy$.

Interchange the order of integration:

$$\left\{ \begin{matrix} 0 \le y \le 1 \\ -\sqrt{1-y} \le x \le \sqrt{1-y} \end{matrix} \right\} \quad \to$$



$$\to \quad \left\{ \begin{matrix} -1 \le x \le 1 \\ 0 \le y \le 1 - x^2 \end{matrix} \right\}$$

Thus,

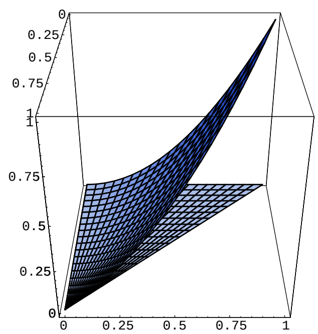
$$\int_0^1 \int_{-\sqrt{1-y}}^{\sqrt{1-y}} e^{3x - x^3}\, dx\, dy = \int_{-1}^1 \int_0^{1-x^2} e^{3x - x^3}\, dy\, dx = \int_{-1}^1 (1 - x^2) e^{3x - x^3}\, dx = \int_0^2 (1 - x)^2 e^u \cdot \frac{du}{3(1 - x^2)} =$$

$$\left[u = 3x - x^3, \quad du = (3 - 3x^2)\, dx = 3(1 - x^2)\, dx, \quad dx = \frac{du}{3(1 - x)^2}; \quad x = 0, \quad u = 0; \quad x = 1, \quad u = 2 \right]$$

$$\frac{1}{3}\int_0^2 e^u\,du = \frac{1}{3}\left[e^u\right]_0^2 = \frac{1}{3}(e^2-1) = 2.12968\ldots. \quad \square$$

---

42. Compute $\iiint_R (6x+4y)\,dV$, where R is the region in the first octant bounded above by $z = y^2$ and bounded on the side by $x+y=1$.



The projection of the region into the x-y-plane (the base of the solid) is the triangle given by the inequalities

$$\left\{\begin{array}{c} 0 \le x \le 1 \\ 0 \le y \le 1-x \end{array}\right\}$$

The top of the region is the parabolic cylinder $z = y^2$. The base of the region is the x-y plane $z = 0$. Thus, the region R is described by the inequalities is given by the inequalities

$$\left\{\begin{array}{c} 0 \le x \le 1 \\ 0 \le y \le 1-x \\ 0 \le z \le y^2 \end{array}\right\}$$

Therefore,

$$\iiint_R (6x+4y)\,dV = \int_0^1 \int_0^{1-x} \int_0^{y^2} (6x+4y)\,dz\,dy\,dx = \int_0^1 \int_0^{1-x} \left[(6x+4y)z\right]_0^{y^2}\,dy\,dx =$$

$$\int_0^1 \int_0^{1-x} (6xy^2+4y^3)\,dy\,dx = \int_0^1 \left[2xy^3+y^4\right]_0^{1-x}\,dx = \int_0^1 (1-2x+2x^3-x^4)\,dx =$$

$$\left[x - x^2 + \frac{1}{2}x^4 - \frac{1}{5}x^5\right]_0^1 = \frac{3}{10}. \quad \square$$

---

43. Compute

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{1+(x^2+y^2+z^2)^{3/2}}\,dz\,dy\,dx.$$

Convert to spherical:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{1+(x^2+y^2+z^2)^{3/2}}\,dz\,dy\,dx = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \frac{\rho^2}{1+\rho^3}\sin\phi\,d\rho\,d\phi\,d\theta =$$

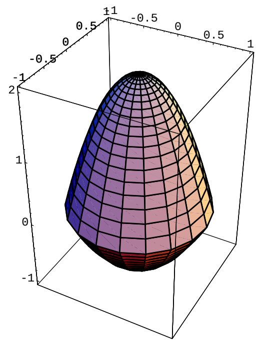
$$\frac{\pi}{2}\int_0^{\pi/2} \sin\phi \left[\frac{1}{3}\ln(1+\rho^3)\right]_0^1 d\phi = \frac{\pi}{6}(\ln 2)\int_0^{\pi/2} \sin\phi\, d\phi = \frac{\pi}{6}(\ln 2)\left[-\cos\phi\right]_0^{\pi/2} =$$

$$\frac{\pi}{6}(\ln 2) = 0.36293\ldots.$$

I did the ρ integral using the substitution $u = 1+\rho^3$. □

---

44. The solid bounded above by $z = 2 - 2x^2 - 2y^2$ and below by $z = x^2 + y^2 - 1$ has density $\rho = 2$. Find the mass and the center of mass.



By symmetry, the center of mass must lie on the z-axis, so $\overline{x} = \overline{y} = 0$.

Find the intersection of the surfaces:

$$\begin{aligned} 2 - 2x^2 - 2y^2 &= x^2 + y^2 - 1 \\ 3x^2 + 3y^2 &= 3 \\ x^2 + y^2 &= 1 \end{aligned}$$

Thus, the projection of the region into the x-y plane is the interior of the unit circle $x^2 + y^2 = 1$. I'll convert to cylindrical. Note that

$$z = 2 - 2x^2 - 2y^2 = 2 - 2r^2 \quad \text{and} \quad z = x^2 + y^2 - 1 = r^2 - 1.$$

The region is

$$\left\{\begin{matrix} 0 \le \theta \le 2\pi \\ 0 \le r \le 1 \\ r^2 - 1 \le z \le 2 - 2r^2 \end{matrix}\right\}$$

The mass is

$$\int_0^{2\pi}\int_0^1\int_{r^2-1}^{2-2r^2} 2r\,dz\,dr\,d\theta = 4\pi\int_0^1 r\,[z]_{r^2-1}^{2-2r^2}\,dr = 4\pi\int_0^1 r(3-3r^2)\,dr = 4\pi\int_0^1 (3r-3r^3)\,dr =$$

$$4\pi\left[\frac{3}{2}r^2 - \frac{3}{4}r^4\right]_0^1 = 3\pi.$$

The moment in the z-direction is

$$\int_0^{2\pi}\int_0^1\int_{r^2-1}^{2-2r^2} 2zr\,dz\,dr\,d\theta = 4\pi\int_0^1 r\left[\frac{1}{2}z^2\right]_{r^2-1}^{2-2r^2} dr = 2\pi\int_0^1 r\left((2-2r^2)^2 - (r^2-1)^2\right) dr =$$

$$2\pi\int_0^1 \left(3r - 6r^3 + 3r^5\right) dr = 2\pi\left[\frac{3}{2}r^2 - \frac{3}{2}r^4 + \frac{1}{2}r^6\right]_0^1 = \pi.$$

Therefore, $\overline{z} = \dfrac{\pi}{3\pi} = \dfrac{1}{3}$. □

---

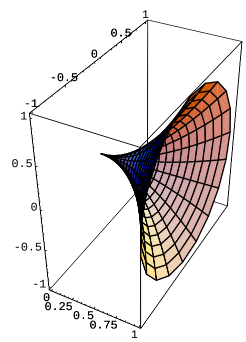
45. (a) Parametrize the surface generated by revolving $y = x^2$ for $0 \le x \le 1$, about the x-axis.

(b) Find the area of the surface.

You may want to make use of the following formula:

$$\int u^2 \sqrt{a^2u^2+1}\, du = \frac{1}{a^3}\left(\frac{au}{4}(a^2u^2+1)^{3/2} - \frac{au}{8}\sqrt{a^2u^2+1} - \frac{1}{8}\ln|\sqrt{a^2u^2+1}+au|\right) + C.$$

(a)



$$x = u, \quad y = u^2 \cos v, \quad z = u^2 \sin v, \quad 0 \le u \le 1, \quad 0 \le v \le 2\pi. \quad \square$$

(b) The normal vector is

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2u\cos v & 2u \sin v \\ 0 & -u^2 \sin v & -u^2 \cos v \end{vmatrix} = (2u^3, -u^2\cos v, -u^2 \sin v).$$

The length of the normal is

$$\|\vec{T}_u \times \vec{T}_v\| = \left(4u^6 + u^4(\cos v)^2 + u^4(\sin v)^2\right)^{1/2} = u^2\sqrt{4u^2+1}.$$

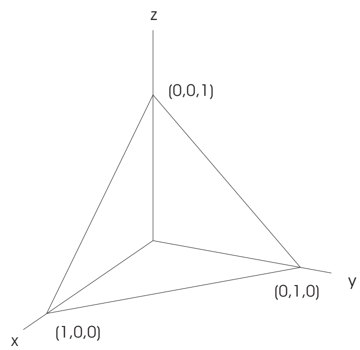
Hence, the area is

$$\int_0^{2\pi}\int_0^1 u^2\sqrt{4u^2+1}\, du\, dv = 2\pi\left[\frac{u}{16}(4u^2+1)^{3/2} - \frac{u}{32}\sqrt{4u^2+1} - \frac{1}{64}\ln|\sqrt{4u^2+1}+2u|\right]_0^1 =$$

$$\pi\left(\frac{1}{8}5^{3/2} - \frac{1}{16}\sqrt{5} - \frac{1}{32}\ln(\sqrt{5}+2)\right) = 3.80972\ldots. \quad \square$$

---

46. A wire is made of the three segments connecting the points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. The density of the wire is $\delta = x + y + z$. Find its mass.



By symmetry, the mass is three times the mass of one of the segments. I will use the segment in the x-y plane: It is the part of the line $x + y = 1$ which goes from $x = 0$ to $x = 1$.

I can parametrize the line by setting $x = t$ with $0 \le t \le 1$, so $y = 1 - t$. This is in the x-y plane, so $z = 0$. Then

$$\vec{\sigma}\,'(t) = (1, -1), \quad \text{so} \quad \|\vec{\sigma}\,'(t)\| = \sqrt{2}.$$

Since $\delta(t) = t + (1 - t) + 0 = 1$, the path integral for this segment is

$$\int_{\vec{\sigma}} \delta\, ds = \int_0^1 1 \cdot \sqrt{2}\, dt = \sqrt{2}.$$

The mass of the whole wire is $3\sqrt{2} = 4.24264\ldots$. $\square$

---

47. Let S be the triangle with vertices $P(1,1,2)$, $Q(2,3,1)$, and $R(-1,2,0)$.

Compute

$$\iint_S (x + 4y + z)\, dS.$$

$$\overrightarrow{PQ} = (1, 2, -1), \quad \overrightarrow{PR} = (-2, 1, -2).$$

The normal vector to the plane containing the triangle is

$$\vec{N} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & -1 \\ -2 & 1 & -2 \end{vmatrix} = (-3, 4, 5).$$

Therefore, the triangle may be parametrized by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} u - 2v + 1 \\ 2u + v + 1 \\ -u - 2v + 2 \end{bmatrix}.$$

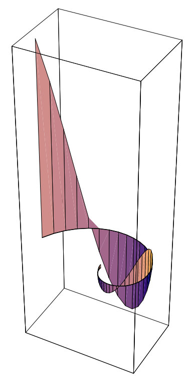
In component form, this is

$$x = u - 2v + 1, \quad y = 2u + v + 1, \quad z = -u - 2v + 2, \qquad 0 \le u \le 1, \quad 0 \le v \le 1 - u.$$

Therefore, $|\vec{N}| = \sqrt{50}$. Since $x + 4y + z = 8u + 7$,

$$\iint_S (x + 4y + z)\, dS = \int_0^1 \int_0^{1-u} (8u + 7)\sqrt{50}\, dv\, du = \sqrt{50} \int_0^1 (8u + 7)(1 - u)\, du = \frac{29\sqrt{50}}{6} = 34.17682\ldots. \quad \square$$

48. Compute $\int_{\vec{\sigma}} f\,ds$, where $f(x,y) = x^2 - y^2$ and $\vec{\sigma}(t) = (e^t \cos 3t, e^t \sin 3t)$, $-1 \le t \le 1$.



First,

$$\vec{\sigma}\,'(t) = (e^t \cos 3t - 3e^t \sin 3t, e^t \sin 3t + 3e^t \cos 3t).$$

Now

$$\begin{aligned}\left(\frac{dx}{dt}\right)^2 &= e^{2t}(\cos 3t)^2 - 2e^{2t}\cos 3t \sin 3t + 9e^{2t}(\sin 3t)^2\\ \left(\frac{dy}{dt}\right)^2 &= e^{2t}(\sin 3t)^2 + 2e^{2t}\cos 3t \sin 3t + 9e^{2t}(\cos 3t)^2\\ \hline \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= e^{2t} + 9e^{2t} = 10e^{2t}\end{aligned}$$

So

$$|\vec{\sigma}\,'(t)| = \sqrt{10e^{2t}} = \sqrt{10}e^t.$$

Therefore, $ds = \sqrt{10}e^t\,dt$.
Next,

$$f(t) = e^{2t}(\cos 3t)^2 - e^{2t}(\sin 3t)^2 = e^{2t}\cos 6t.$$

Hence,

$$\int_{\vec{\sigma}} f\,ds = \int_{-1}^{1} e^{2t}\cos 6t \cdot \sqrt{10}e^t\,dt = \frac{\sqrt{10}}{15}\left(e^3\cos 6 + 2e^3\sin 6 - e^{-3}\cos 6 + 2e^{-3}\sin 6\right) = 1.68348\ldots. \quad \square$$

49. Let

$$\vec{\sigma}(t) = \left(te^{t-1}, t^3, \sin\left(\frac{\pi t}{2}\right)\right), \quad 0 \le t \le 1.$$

Compute

$$\int_{\vec{\sigma}} (y - z^2)\,dx + (x - 2y + 2yz)\,dy + (y^2 + 2z - 2xz)\,dz.$$

Let $\vec{F} = (y - z^2, x - 2y + 2yz, y^2 + 2z - 2xz)$. Then

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y - z^2 & x - 2y + 2yz & y^2 + 2z - 2xz \end{vmatrix} = (0,0,0).$$

Hence, the field is conservative. A potential function f must satisfy

$$\frac{\partial f}{\partial x} = y - z^2, \quad \frac{\partial f}{\partial y} = x - 2y + 2yz, \quad \frac{\partial f}{\partial z} = y^2 + 2z - 2xz.$$

Integrate the first equation with respect to x:

$$f = \int (y - z^2)\, dx = yx - xz^2 + C(y, z).$$

Since the integral is with respect to x, the arbitrary constant may depend on y and z. Differentiate with respect to y:

$$x - 2y + 2yz = \frac{\partial f}{\partial y} = x + \frac{\partial C}{\partial y}.$$

$\dfrac{\partial C}{\partial y} = -2y + 2yz$, so

$$C = \int (-2y + 2yz)\, dy = -y^2 + y^2 z + D(z).$$

Since the integral is with respect to y, the arbitrary constant may depend on z. Now

$$f = yx - xz^2 - y^2 + y^2 z + D(z).$$

Differentiate with respect to z:

$$y^2 + 2z - 2xz = \frac{\partial f}{\partial z} = -2xz + y^2 + \frac{dD}{dz}.$$

$\dfrac{dD}{dz} = 2z$, so $D = z^2 + E$. At this point, E is a numerical constant; since the derivative of a number is 0, and since I only need some potential function, I may take $E = 0$. Then $D = z^2$, so

$$f = yx - xz^2 - y^2 + y^2 z + z^2.$$

Now use path independence. The endpoints of the path are

$$\vec{\sigma}(0) = (0, 0, 0), \quad \vec{\sigma}(1) = (1, 1, 1).$$

Hence,

$$\int_{\vec{\sigma}} (y - z^2)\, dx + (x - 2y + 2yz)\, dy + (y^2 + 2z - 2xz)\, dz = f(1, 1, 1) - f(0, 0, 0) = 1. \quad \square$$

---

50. Let

$$\vec{F} = \left(\frac{x}{x^2 + y^2 + z^2 + 1}, \frac{y}{x^2 + y^2 + z^2 + 1}, \frac{z}{x^2 + y^2 + z^2 + 1}\right).$$

Let $\vec{\sigma}(t)$ be *any* path from *any* point on the sphere $x^2 + y^2 + z^2 = 1$ to *any* point on the sphere $x^2 + y^2 + z^2 = 5$. Compute $\displaystyle\int_{\vec{\sigma}} \vec{F} \cdot \overrightarrow{ds}$.

Let P be a point on the sphere $x^2 + y^2 + z^2 = 1$ and let Q be a point on the sphere $x^2 + y^2 + z^2 = 5$. If $f(x, y, z) = \dfrac{1}{2} \ln(x^2 + y^2 + z^2 + 1)$, then

$$\nabla f = \left(\frac{x}{x^2 + y^2 + z^2 + 1}, \frac{y}{x^2 + y^2 + z^2 + 1}, \frac{z}{x^2 + y^2 + z^2 + 1}\right).$$

Hence, by path independence,

$$\int_{\vec{\sigma}} \vec{F} \cdot \vec{ds} = f(Q) - f(P) = \frac{1}{2}\ln(5+1) - \frac{1}{2}\ln(1+1) = \frac{1}{2}(\ln 6 - \ln 2) = 0.54930\ldots.$$

Explanation: Since Q is on $x^2+y^2+z^2=5$, for this point $\ln(x^2+y^2+z^2+1) = \ln(5+1)$. Likewise, $\ln(x^2+y^2+z^2+1) = \ln(1+1)$ for P, because P is on $x^2+y^2+z^2=1$. □

---

51. (a) Let $\vec{\sigma}$ denote the circle $x^2+y^2=2x$, traversed counterclockwise. Compute

$$\int_{\vec{\sigma}} (2xy - x^2)\,dx + (xy + x^2)\,dy.$$

(b) The vector field in the integral is not conservative, but the integral around the closed curve $\vec{\sigma}$ is 0. Is there anything wrong with this?

(a) Let R be the circular region enclosed by the curve. By Green's Theorem,

$$\int_{\vec{\sigma}} (2xy - x^2)\,dx + (xy + x^2)\,dy = \iint_R (y + 2x - 2x)\,dx\,dy = \iint_R y\,dx\,dy.$$

Convert to polar.

$$\begin{aligned} x^2+y^2 &= 2x \\ r^2 &= 2r\cos\theta \\ r &= 2\cos\theta \end{aligned}$$

The circle is $r = 2\cos\theta$, and it is traced out once as θ goes from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. Hence, the region R is

$$\left\{ \begin{matrix} -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ 0 \le r \le 2\cos\theta \end{matrix} \right\}$$

Hence, the integral becomes

$$\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r\sin\theta \cdot r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2 \sin\theta\,dr\,d\theta =$$

$$\frac{8}{3}\int_{-\pi/2}^{\pi/2} (\cos\theta)^3(\sin\theta)\,d\theta = \frac{8}{3}\left[-\frac{1}{4}(\cos\theta)^4\right]_{-\pi/2}^{\pi/2} = 0.$$

Here's the work for the θ integral:

$$\int (\cos\theta)^3(\sin\theta)\,d\theta = \int u^3(\sin\theta)\cdot\frac{du}{-\sin\theta} = -\int u^3\,du =$$

$$\left[u = \cos\theta, \quad du = -\sin\theta\,d\theta, \quad d\theta = \frac{du}{-\sin\theta}\right]$$

$$-\frac{1}{4}u^4 + c = -\frac{1}{4}(\cos\theta)^4 + c. \quad \Box$$

(b) No. *If* the field is conservative, then the integral around every closed curve should be 0. If the field is *not* conservative, then the integral around a closed curve *may or may not* be 0. □

---

52. Consider the ellipse

$$x = a\cos t, \quad y = b\sin t, \qquad 0 \le t \le 2\pi.$$

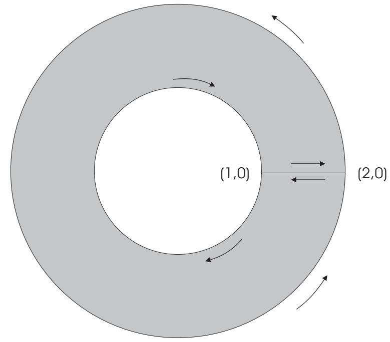
Use Green's Theorem to show that the area of the ellipse is πab.

$\dfrac{dy}{dt} = b\cos t$, so the area is

$$\int_C x\,dy = \int_0^{2\pi} ab(\cos t)^2\,dt = ab\int_0^{2\pi} \frac{1}{2}(1+\cos 2t)\,dt = ab\left[\frac{1}{2}t + \frac{1}{4}\sin 2t\right]_0^{2\pi} = \pi ab. \quad \Box$$

---

53. Let $\vec{\sigma}$ be the path which starts at $(2,0)$, goes around the circle $x^2+y^2=4$ in the counterclockwise direction, traverses the segment from $(2,0)$ to $(1,0)$, goes around the circle $x^2+y^2=1$ in the clockwise direction, and traverses the segment from $(1,0)$ to $(2,0)$. Compute $\displaystyle\int_{\vec{\sigma}} -y\,dx + x\,dy$.



Let R denote the ring-shaped area between the two circles. By Green's theorem,

$$\int_{\vec{\sigma}} -y\,dx + x\,dy = \iint_R (1-(-1))\;dx\,dy = 2\cdot(\text{area of } R).$$

The area of R is the area of the outer circle minus the area of the inner circle, or $4\pi - \pi = 3\pi$. Hence,

$$\int_{\vec{\sigma}} -y\,dx + x\,dy = 6\pi = 18.84955\ldots. \quad \Box$$

---

54. Compute the circulation of $\vec{F} = (yz, xz, -xy)$ counterclockwise (as viewed from above) around the triangle with vertices $A(1,2,1)$, $B(2,1,4)$, and $C(-3,1,1)$.

$\overrightarrow{AB} = (1,-1,3)$ and $\overrightarrow{AC} = (-4,-1,0)$, so the triangle may be parametrized by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ -1 & -1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

The limits are

$$\left\{\begin{matrix} 0 \le u \le 1 \\ 0 \le v \le 1-u \end{matrix}\right\}$$

Reason: If I use $0 \le u \le 1$, $0 \le v \le 1$, the input is a square and the output is a parallelogram (four-sided figure to four-sided figure). Since I only want a triangle — half the parallelogram — I only feed half the square into the transformation (three-sided figure to three-sided figure).

In component form, this is

$$x = u - 4v + 1, \quad y = -u - v + 2, \quad z = 3u + 1.$$

The normal is

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 3 \\ -4 & -1 & 0 \end{vmatrix} = (3, -12, -5).$$

I need the upward normal (the boundary is traversed "counterclockwise as viewed from above"), so I negate this vector to get $(-3, 12, 5)$.

Now

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & xz & -xy \end{vmatrix} = (-2x, 2y, 0).$$

So

$$\operatorname{curl} \vec{F} \cdot (\vec{T}_u \times \vec{T}_v) = 54 - 18u - 48v.$$

By Stokes' theorem, the circulation of $\vec{F}$ around the boundary of the triangle is

$$\int_0^1 \int_0^{1-u} (54 - 18u - 48v)\, dv\, du = \int_0^1 \left[(54 - 18u)v - 24v^2\right]_0^{1-u} du =$$

$$\int_0^1 \left(54 - 18u)(1 - u) - 24(1 - u)^2\right) du = \int_0^1 \left(54 - 72u + 18u^2 - 24(1 - u)^2\right) du =$$

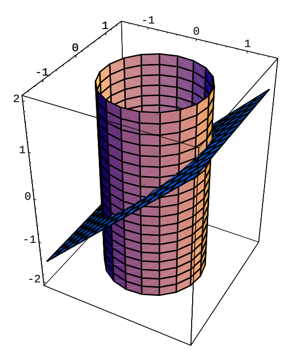
$$\left[54u - 36u^2 + 6u^3 + 8(1 - u)^3\right]_0^1 = 16. \quad \square$$

---

55. Let $\vec{\sigma}$ be the curve of intersection of the plane $z = x$ and the cylinder $x^2 + y^2 = 1$, traversed counterclockwise as viewed from above. Compute the circulation of $\vec{F} = (x^2y^3, 1, z)$ around $\vec{\sigma}$:

(a) Directly, by parametrizing the curve and computing the line integral.

(b) Using Stokes' theorem.



(a) First, I'll compute the circulation directly. To parametrize the curve of intersection, project it into the x-y plane. I get $x^2 + y^2 = 1$, which I can parametrize by $x = \cos t$, $y = \sin t$. Now $z = x$, so $z = \cos t$. Thus, the curve is

$$\vec{\sigma}(t) = (\cos t, \sin t, \cos t), \quad 0 \le t \le 2\pi.$$

Now

$$\vec{F}(t) = ((\cos t)^2 (\sin t)^3, 1, \cos t),$$

$$\vec{\sigma}\,'(t) = (-\sin t, \cos t, -\sin t),$$

$$\vec{F}(t)\cdot\vec{\sigma}\,'(t) = -(\cos t)^2(\sin t)^4 + \cos t - \sin t\cos t.$$

The circulation is

$$\int_0^{2\pi} \left(-(\cos t)^2(\sin t)^4 + \cos t - \sin t\cos t\right)\, dt = -\frac{\pi}{8}.$$

Note that you can integrate $(\cos t)^2(\sin t)^4$ by using the double angle formulas, but it's a little messy.
□

(a) Next, I'll use Stokes' theorem. The surface is the plane $z = x$, for which the normal is

$$\vec{N} = \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right) = (-1, 0, 1).$$

The z-component is positive, so this is the upward normal, consistent with the orientation of the curve. Next,

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y^3 & 1 & z \end{vmatrix} = (0, 0, -3x^2y^2).$$

Hence,

$$\operatorname{curl}\vec{F}\cdot\vec{N} = -3x^2y^2.$$

The projection of the surface into the x-y plane is the interior of the unit circle. I'll convert to polar. The projection is

$$\left\{\begin{array}{c} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{array}\right\}$$

Moreover,

$$\operatorname{curl}\vec{N}\cdot\vec{N} = -3r^4(\cos\theta)^2(\sin\theta)^2.$$

So by Stokes' theorem, the circulation is

$$\int_0^{2\pi}\int_0^1 -3r^4(\cos\theta)^2(\sin\theta)^2 r\,dr\,d\theta = \int_0^{2\pi} (\cos\theta)^2(\sin\theta)^2\left[-\frac{1}{2}r^6\right]_0^1 d\theta =$$

$$-\frac{1}{2}\int_0^{2\pi}(\cos\theta)^2(\sin\theta)^2\,d\theta = -\frac{1}{8}\int_0^{2\pi}(1+\cos 2\theta)(1-\cos 2\theta)\,d\theta = -\frac{1}{8}\int_0^{2\pi}\left(1-(\cos 2\theta)^2\right)d\theta =$$

$$-\frac{1}{8}\int_0^{2\pi}(\sin 2\theta)^2\,d\theta = -\frac{1}{16}\int_0^{2\pi}(1-\cos 4\theta)\,d\theta = -\frac{1}{16}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{2\pi} = -\frac{\pi}{8}. \quad \square$$

---

56. Let R be the solid region in the first octant cut off by the sphere $x^2 + y^2 + z^2 = 1$. Compute the flux out through the boundary of R of the vector field

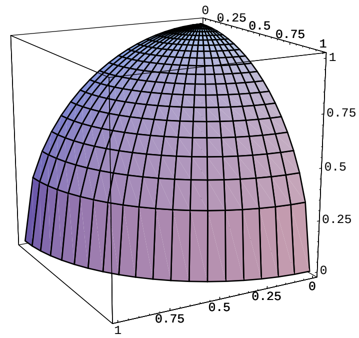
$$\vec{F} = \left(x^3 + yz, y^3 - xz, z^3 + 2xy\right).$$

By the Divergence Theorem,

$$\iint_{\partial R} \vec{F}\cdot d\vec{S} = \iiint_R \operatorname{div}\vec{F}\,dV.$$

I'll use spherical coordinates.

$$\operatorname{div}\vec{F} = 3x^2 + 3y^2 + 3z^2 = 3\rho^2.$$



The region is

$$\left\{\begin{array}{c} 0 \le \theta \le \dfrac{\pi}{2} \\ 0 \le \rho \le 1 \\ 0 \le \phi \le \dfrac{\pi}{2} \end{array}\right\}$$

Therefore,

$$\iint_{\partial R} \vec{F} \cdot d\vec{S} = \int_0^{\pi/2} \int_0^1 \int_0^{\pi/2} 3\rho^2 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{3\pi}{2} \int_0^1 \rho^4 \left[-\cos\phi\right]_0^{\pi/2} \, d\rho = \frac{3\pi}{2} \int_0^1 \rho^4 \, d\rho =$$

$$\frac{3\pi}{2} \left[\frac{1}{5}\rho^5\right]_0^1 = \frac{3\pi}{10} = 0.94247 \ldots. \quad \square$$

The best thing for being sad is to learn something. - Merlyn, in T. H. White's *The Once and Future King*